Qualifying Exam - Analysis - Fall 2016

Justify your answers to all problems.

1. Assume $f, f_j \subset L^2([0,1])$ for $j = 1, 2, \ldots$ and $||f_j - f||_{L^2} \to 0$. Prove there exists a subsequence $\{f_{j'}\} \subset \{f_j\}$ such that $f_{j'} \to f$ a.e.

2. Suppose A is a Lebesgue measurable set in $\mathbb{R}$ with $m(A) > 0$. Does there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that the complement of $\bigcup_{n=1}^{\infty}(A + x_n)$ in $\mathbb{R}$ has measure 0? Justify your answer. (We define $A + x_n = \{a + x_n \in \mathbb{R} : a \in A\}$.)

3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Determine if the following statements are true or false. If true, provide a proof. If false, provide a counter example.

(a) A sequence $\{f_n\}$ in $\mathcal{H}$ with $||f_n|| = 1$ for all n has a subsequence that converges in $\mathcal{H}$.

(b) A sequence $\{f_n\}$ in $\mathcal{H}$ with $||f_n|| = 1$ for all n has a subsequence that converges weakly in $\mathcal{H}$.

4. Prove that if a sequence of harmonic functions on the open disk converges uniformly on compact subset of the disk, then the limit is harmonic.

5. Let f be a one-to-one analytic function defined on the unit disk D centered at the origin and $f(0) = 0$. Show that the function $g(z) = \sqrt{f(z^2)}$ has a single-valued branch and is also one-to-one.

6. Let $U \subset \mathbb{C}$ be an open set containing the closure $\overline{D}$ of a unit disk. If a sequence $\{f_n : U \to \mathbb{C}\}$ of holomorphic functions converges uniformly to f on compact subsets of U, then show that there exists an integer N such that f and f_n have the same number of zeros in D for $n \geq N$.