# Qualifying Exam - Analysis-Fall 2017

12:30-3:30pm, Sept 8, 2017

1. Let $f_n$ be a sequence of continuous functions on $\mathbb{R}$ satisfying $0 \leq f_n \leq f_{n+1} \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that if $f$ is continuous at $x$, then for any $\epsilon > 0$ there exist $\delta$ and $N$ such that $|f_n(y) - f_n(x)| < \epsilon$ whenever $|y - x| < \delta$ and $n > N$.

2. Let $f \in L^p(\mathbb{R}^n)$. Show that

$$\lim_{h\to 0} \|f(x-h) - f(x)\|_{L^p} = 0.$$

3. For a Radon measure $\mu$, with $\int_{\mathbb{R}^n} d\mu = C$. Prove that for all $\epsilon > 0$, there exists a set $E_\epsilon \subset \mathbb{R}^n$ s.t. $\mathcal{M}^1(E_\epsilon) := \inf_{E_\epsilon \subset \cup B_i}\{\sum_i \operatorname{diam} B_i\} < 10\epsilon$ and for any $x \notin E_\epsilon$, $r > 0$

$$\int_{B_r(x)} d\mu \leq \frac{Cr}{\epsilon}.$$

(Hint: use Vitali covering lemma.)

4. Let $f(z)$ be a holomorphic function on $D := \{z \in \mathbb{C} : |z| < 1\}$, $|f(z)| < 1$, $f(\alpha) = 0$ for some $|\alpha| < 1$. Show that for $z \in D$

$$|f(z)| \leq |\frac{z-\alpha}{1-\bar{\alpha}z}|.$$

5. Let $f$ be an entire function $|Re(f(z))| \leq C(1+|z|)^p$ for some $p > 0$, $C > 0$. Show that $f$ is a polynomial.

6. Let $u$ be a subharmonic function defined on $\mathbb{C}$. Let $M(r) := \max_{|z|=r} u(z)$. Prove that

$$u(z) \leq \frac{\log r_2 - \log|z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log|z| - \log r_1}{\log r_2 - \log r_1} M(r_2)$$

for $0 < r_1 \leq |z| \leq r_2$.