Analysis Qualifying Exam, Spring 2004

Instructions: Do all problems. Show all details in your solutions. Unless stated otherwise, you may cite any of the theorems mentioned in the syllabus.

1. Consider the sequence of functions $g_n(x) = [\sin(nx)]^2$ on $[0, 2\pi]$. Define each of the following notions of convergence and determine whether the sequence converges in that sense; if so, determine the limit:

a) Converges pointwise

b) Converges strongly in L^1

c) Converges weakly in L^1

2. Consider the set of positive continuous periodic functions f on $[0, 2\pi]$ satisfying $\frac{1}{2\pi}\int_0^{2\pi} f\,d\theta = 1$. What is the largest possible value of $\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log f\,d\theta\right)$ for such functions? Prove that your answer is correct.

3. Let $\alpha > 1/2$ and consider $f_\alpha(x) = \int_{\mathbb{R}} (1+\xi^2)^{-\alpha} e^{2\pi i x\xi}\,d\xi$.
Without doing the integration, determine, for each α, which of the following properties holds for f_α, and prove that your answer is correct:

a) i) $\lim_{|x|\to\infty} |f_\alpha(x)| = 0$, ii) $f_\alpha \in L^2(\mathbb{R})$.

b) *Without appealing to the properties of the Fourier transform*, show that
i) $f_\alpha \in C(\mathbb{R})$, ii) f_α is bounded on $\mathbb{R}$.

4. In problem #3, take $\alpha = 1$. Calculate $f_1(x)$, as defined in #3, by the method of residues.

5. a) Let $f(z)$ be complex analytic in the disc $|z| < \pi$. Assume that the only zero of f in the closed unit disc $\overline{D} = \{z : |z| \le 1\}$ is a simple zero at the origin. Let C be the unit circle, oriented counterclockwise. Evaluate

$$\int_C \frac{dz}{f(z)},$$

in the sense that no integration symbols should appear in the answer.

b) Let f be as in part a), *except* assume that f has a 2nd-order (i.e., double) zero at the origin. Verify or give a counterexample:

$$\text{Assertion:} \quad \int_C \frac{dz}{f(z)} = 0.$$

6. Let $f(z)$ be holomorphic in an open set containing the closed unit disc $\overline{D}$. Suppose that $|f(z)| < 1$ for all z on the unit circle. Show that there is exactly one point $z \in D$ (the interior of $\overline{D}$) for which $f(z) = z$.

7. Determine a one-to-one complex analytic mapping f, **other than** $f(z) = z$, that takes D (notation as above) onto itself and satisfies $f(\frac{1}{3}) = \frac{1}{3}$.