ANALYSIS QUALIFYING EXAM
SPRING 2005

Notation:

$\mathcal{C}^\infty(\mathbb{R})$: complex-valued $\mathcal{C}^\infty$ functions on $\mathbb{R}$.
$\mathcal{C}_c^\infty(\mathbb{R})$: compactly supported functions in $\mathcal{C}^\infty(\mathbb{R})$
$L^p(\mathbb{R}),\ L^p([0,1])$: L^p functions with respect to Lebesgue measure on $\mathbb{R}$, $[0,1]$, respectively
$\widehat{f}$: Fourier transform of f
$D = \{z \in \mathbb{C} : |z| < 1\}$
$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

Do all 8 problems. Show all work. In each solution, state which theorems from 110.605 and 110.607 you are applying and verify that the hypotheses are satisfied.

(1) Let $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$.
  (a) Is $\widehat{f} \in \mathcal{C}^\infty(\mathbb{R})$? Prove that your answer is correct.
  (b) Show that $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

(2) Suppose that $f \in L^1[0,1]$ and let $g(x) = \displaystyle\int_x^1 \frac{f(t)}{t} dt$. Show that $g \in L^1[0,1]$ and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

(3) Prove or find a counterexample to each of the following statements:
  (a) $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$;
  (b) $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$;
  (c) $L^2([0,1]) \subset L^1([0,1])$;
  (d) $L^1([0,1]) \subset L^2([0,1])$;

(4) Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H.
  (a) Show that $e_n \to 0$ weakly. (Explain what weak convergence means.)
  (b) Show that e_n does not tend to zero strongly. (Explain what strong convergence means.)
  (c) Let $v_n = \frac{1}{n}\sum_{j=1}^n e_j$. Show that $v_n \to 0$ strongly.

(5) Do there exist functions $f \in \mathcal{C}_c^\infty(\mathbb{R})$ such that f is not identically zero and $\widehat{f} \in \mathcal{C}_c^\infty(\mathbb{R})$? If so, find one. If not, prove that none exist.
Hint: Consider $\widehat{f}(\xi)$ for $\xi \in \mathbb{C}$.

(6) Use residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)} .$$

(7) Find a bijective holomorphic map f from the quadrant

$$Q = \{x + iy \in \mathbb{C} : x > 0,\ y > 0\}$$

onto the unit disk D in $\mathbb{C}$ with $f(1+i) = 0$.

(8) Let U be an open set in $\mathbb{C}$ containing the closed unit disk $\overline{D}$. Suppose f is a meromorphic function on U such that $f(\partial D) \subset \mathbb{R}^+$. (In particular, f has no zeros or poles on ∂D.) Show that f has the same number of zeros as poles in D (counting multiplicities).