ANALYSIS QUALIFYING EXAM
MAY 2006

All problems are equally weighted. Time: 3 hours.

Show all work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Part I. Complex Analysis. Do 5 out of the following 6 problems.

1. Let P be a point in an open set U in $\mathbb{C}$, and suppose that f is a meromorphic function on U with a pole at P. Prove that there is no holomorphic function $g : U \setminus \{P\} \to \mathbb{C}$ such that $e^{g(z)} = f(z)$ for all $z \in U \setminus \{P\}$.

2. How many zeros does the polynomial

$$z^7 - 4\,z^3 + z - \tfrac{1}{2}$$

have in the unit disk $\{|z| < 1\}$? How many zeros does it have in the disk $\{|z| < 2\}$ of radius 2? Justify your answers.

3. Find all entire functions f such that $|f(z)| \leq |z|^{3/2}$ whenever $|z| \geq 1$. Give explicit formulas for the functions and give a proof for your answer. (An entire function is a holomorphic function on $\mathbb{C}$.)

4. Let $f_n : D \to (-\infty, 1)$, $n = 1, 2, \ldots$, be an increasing sequence of harmonic functions on the unit disk D such that $f_n(0) \to 1$ as $n \to \infty$. (I.e., $f_n(z) \leq f_{n+1}(z) < 1$, $\forall n \geq 1$.) Prove that $f_n(z) \to 1$ as $n \to \infty$, for all $z \in D$.

5. Let H denote the upper half plane $\{z \in \mathbb{C} : \text{Im } z > 0\}$. Suppose that $f : H \to H$ is holomorphic, and $f(3 + 17i) = 3 + 17i$. What is the maximum possible value of $f'(3+17i)$. Give a reason for your answer (and try not to do any lengthy computations).

6. Find all the poles of the function

$$f(z) = \frac{e^{\pi z}}{(z^2 + 1)^2} \ .$$

Determine the residue of f at each pole.

Part II. Real Analysis. Do 5 out of the following 6 problems.

7. Quickies:

a) Give an example of a function that is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$.

b) Give an example of a function that is in $L^1((0, 1))$ but not in $L^2((0, 1))$.

8. Prove that any function $f \in L^1(I) \cap L^2(I)$ for any interval $I \subset \mathbb{R}$ must be in $L^p(I)$ for all p between 1 and 2.

9. Suppose that f is in $L^1(\mathbb{R})$. Prove directly (i.e., without citing properties of the Fourier transform) that the function

$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-ixt} f(x)\, dx$$

is uniformly continuous and $\widehat{f}(t) \to 0$ as $t \to \infty$.

10. Suppose that f is in $L^1(\mathbb{R})$. Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| = 0\,.$$

11. Suppose that f_n is a sequence of functions in $L^2([0,1])$ that converges weakly to a function $f \in L^2([0,1])$. Either prove that $\limsup_{n\to\infty} ||f_n||_{L^2([0,1])} < \infty$ or give a counter-example.

12. Let f_j be an orthonormal sequence in $L^2([0,1])$. Prove that

$$S_n = \frac{1}{n} \sum_{j=1}^{n} f_j$$

converges to zero a.e.