PROBLEMS FOR ANALYSIS QUALIFYING EXAM
SPRING 2007

(1) How many zeros does the polynomial $z^6 - 2z^5 + 7z^4 + z^3 - z + 1$ have in the open unit disc $D = \{z : |z| < 1\}$?

(2) Calculate the integral $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, where $0 < a < 1$.

(3) Let $f : D \to D$ be a holomorphic map of the unit disc with $f(0) = 0$, and suppose that f is *not* a rotation (a rotation is a map $r_\theta(z) = e^{i\theta}z$). Let $w \in D$ and consider the sequence $\{w_n\}$ defined by $w_{n+1} = f(w_n)$. Show: $\lim_{n\to\infty} w_n = 0$.

(4) Does there exist a surjective holomorphic map $f : D \to \mathbb{C}$ from the unit disc to the whole complex plane? Prove that your answer is correct.

(5) For which p's is the function $1/x$ in $L^p(0,\infty)$?

(6) Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of L^4 functions with $\int f_n^4 \le 1$ for every n and so that $\lim_{n\to\infty} \int |f_n| = 0$. Show that f_n goes to 0 weakly in L^4.

(7) Suppose that f_n is a sequence of functions in $L^2(\mathbb{R})$ that converges weakly in L^2 to a function $f \in L^2(\mathbb{R})$. Is it possible to have

$$\lim_{n\to\infty} ||f_n||_{L^2} = \infty\,?$$

(8) Suppose that $f \in L^1(\mathbb{R})$ and $\widehat{f}(z) = \int_{\mathbb{R}} e^{-ixz} f(x)\,dx$. Show that f and $\widehat{f}$ cannot both have compact support (except if f is identically zero).