PROBLEMS FOR ANALYSIS QUALIFYING EXAM
SPRING 2008

Do all eight problems. Show all work and state any theorems you are using. Time: 3 hours.

1) Let E, F be two Lebesgue measurable subsets of $\mathbb{R}$ of finite measure, and let χ_E, χ_F be their respective characteristic functions.

a) Show that the convolution $\chi_E * \chi_F$ defined by

$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y)\chi_F(x-y)\,dy$$

is a continuous function.

b) Show that

$$n\big(\chi_E * \chi_{[0,1/n]}\big) \to \chi_E$$

as $n \to \infty$ pointwise almost everywhere.

2) Consider $L^\infty([0,1])$.

a) If f belongs to this space prove that

$$\lim_{p\to\infty} \left(\int_0^1 |f|^p\,dx\right)^{1/p} = \|f\|_\infty.$$

b) Give an example showing that this is false if we replace $L^\infty([0,1])$ by $L^\infty(\mathbb{R})$.

3) Assume that f is a continuously differentiable 2π periodic function on $\mathbb{R}$. Show that the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

is absolutely convergent for every t (here $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt$).

4) Let ℓ^2 be the space of all square-summable sequences of complex numbers, and let $T : \ell^2 \to \ell^2$ be a linear operator. Let e_n be the sequence

$$e_n = (0\,0\cdots0\,1\,0\cdots),$$

where 1 is in the n-th position. Let $a_{mn} = \langle Te_m, e_n\rangle$ be the "matrix coefficients" of T.

a) Assume that $\sum_{n,m=1}^{\infty} |a_{mn}|^2 < \infty$. Show that T is a bounded operator on ℓ^2.

b) Assume instead that $\sup\{\,|a_{mn}| : 1 \le n,m < \infty\}$ is finite. Must T be bounded? Explain.

5) Prove the following statement: If f and g are entire functions, $g(z) \neq 0$ and $|f(z)| \le |g(z)|$ for all $z \in \mathbf{C}$, then $f(z) = Cg(z)$ for some constant C.

6) Let $D = \{z \in \mathbf{C} : |z| < 1\}$ and P and Q be distinct points in D. Prove the following statement: If f and g are conformal (or equivalently biholomorphic) self-maps of D, $f(P) = g(P)$ and $f(Q) = g(Q)$, then $f \equiv g$.

7) Let $U \subset \mathbf{C}$ be an open set, $P \in U$ and f a holomorphic function defined on U so that $f(P) = f'(P) = 0$. Use the Argument Principle to prove the following statement: There exists $\delta > 0$ so that if $0 < |Q| < \delta$, then $f^{-1}(Q)$ contains at least two points.

8) Let $U \subset \mathbf{C}$ be an open set and $P \in U$. Let $\mathcal{F}$ be a family of holomorphic functions from U into the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ that take P to 0.

(a) Show that $\sup\{|f'(P)| : f \in \mathcal{F}\} < \infty$.

(b) Show that there exists a sequence $\{f_n\} \subset \mathcal{F}$ and a holomorphic function $f_0 : U \to D$ so that $\{f_n\}$ converges uniformly to f_0 on every compact subset of U and $f_0'(P) = \sup\{|f'(P)| : f \in \mathcal{F}\}$.