ANALYSIS QUALIFYING EXAM
MAY 2009

Do all 8 problems. All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

1. Find all meromorphic functions f on $\mathbb{C}$ such that

$$|f(z)| \leq \frac{\log(2+|z|^2)}{|z|} \qquad \text{for all} \quad z \neq 0\,.$$

   Give explicit formulas for the functions and give a proof for your answer.

2. How many solutions does the equation

$$z + e^{-z} = 2 + i$$

   have in the half-plane $\operatorname{Re} z > 0$? Prove that your answer is correct.

3. Let $f_n : U \to \mathbb{C}$, $n = 1, 2, 3, \ldots$, be a sequence of holomorphic functions such that $f_n^{-1}(0) = \{c_n\}$, where $c_n \in U$, and U is a connected open set. Suppose that $f_n \to f_0$ uniformly, where f_0 is not constant.

   a) Prove that f_0 has at most one zero in U.

   b) Can f_0 have no zeros? If so, give a necessary and sufficient condition on the c_n for this to happen.

4. Let $f(x) = \dfrac{1}{x^2+1}$. Use a contour integral consisting of the interval $[-R, R] \subset \mathbb{R}$ and a semicircle of radius R to compute the Fourier transform

$$\widehat{f}(1) = \int_{\mathbb{R}} f(x) e^{-ix}\, dx\ .$$

   Show that the contour integral converges to your answer as $R \to +\infty$.

5. Let $f, g \in L^2(\mathbb{R})$ be two square-integrable functions on $\mathbb{R}$ (with the usual Lebesgue measure). Show that the convolution

$$f * g(x) = \int_{\mathbb{R}} f(y) g(x-y)\, dy$$

   of f and g is a bounded continuous function on $\mathbb{R}$.

6. Let $\mathbb{R}/\mathbb{Z}$ be the unit circle with the usual Lebesgue measure. For each $n = 1, 2, 3, \ldots$ let $K_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}_+$ be a nonnegative integrable function such that $\int_{\mathbb{R}/\mathbb{Z}} K_n(t)\,dt = 1$ and $\lim_{n\to\infty} \int_{\varepsilon \le |t| \le 1/2} K_n(t)\,dt = 0$ for every $0 < \varepsilon < 1/2$, where we identify $\mathbb{R}/\mathbb{Z}$ with $(-1/2, 1/2]$ in the usual way. (Such a sequence of K_n are called *approximations to the identity*.) Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be continuous, and define the convolutions $f * K_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$f * K_n(x) = \int_{\mathbb{R}/\mathbb{Z}} f(x - t) K_n(t)\,dt.$$

Show that $f * K_n$ converges uniformly to f.

7. Fix $1 \le p < \infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions $f_n : [0, 1] \to \mathbb{C}$. Suppose there exists $f \in L^p([0, 1])$ such that $f_n \to f$ in L^p, that is,

$$\int_{[0,1]} |f_n(x) - f(x)|^p\,dx \to 0.$$

a) Show that $f_n \to f$ in measure, that is,

$$\lim_{n\to\infty} \mu(\{x \in [0, 1] : |f_n(x) - f(x)| \ge \varepsilon\}) = 0$$

for all $\varepsilon > 0$. (Here μ =Lebesgue measure.)

b) Show that there is a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ almost everywhere.

8. Consider $[0, 1]$ with Lebesgue measure. Let $f \in L^\infty([0, 1])$ and define

$$a_n = \int_{[0,1]} |f|^n\,dx.$$

Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty.$$