ANALYSIS QUALIFYING EXAM
MAY 2010

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

1. Let f be a holomorphic function on the punctured disk
$$U := \{z \in \mathbb{C} : 0 < |z| < 1\} .$$
Suppose that $|f(z)| \leq |z|^{-1/2}$ for all $z \in U$. Prove that f has a removable singularity at 0.

2. Find all possible values of
$$\int_\gamma \frac{e^{\pi z}}{(z-1)(z-i)^2}\, dz$$
where γ ranges over all simple closed smooth curves contained in $\mathbb{C} \setminus \{1, i\}$. (A simple closed curve is a closed curve that does not intersect itself; i.e., it is a homeomorphic image of the circle.)

You do not need to give a proof for your answer to this problem, but show all your work.

3. Let $\mathcal{O}(D)$ denote the space of holomorphic functions on the unit disk D and let
$$\mathcal{H} = \mathcal{O}(D) \cap L^2(D) = \left\{ f \in \mathcal{O}(D) : \int_D |f|^2 \, dx\, dy < +\infty \right\} .$$
a) Show that for all compact sets $K \subset D$, there is a constant $C_K \in \mathbb{R}^+$ such that
$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{L^2(D)} .$$
b) Show that $\mathcal{H}$ is a closed subspace of $L^2(D)$ and hence is a Hilbert space.

4. Let h be a harmonic function on the domain
$$U := \{z \in \mathbb{C} : |z| > 1\} .$$
Show that there exists a constant $c \in \mathbb{R}$ and a holomorphic function f on U such that $\text{Re} f(z) = h(z) + c \log |z|$ for all $z \in U$.

Part II. Real Analysis.

5. Let $f_j \in L^2(\mathbb{R}^n)$, and $\widehat{f_j}$ denote its Fourier transform for $j = 1, 2, 3 \dots$. Suppose that $f_j \to f$ in L^2 and that there is a finite constant M so that

$$\|f_j\|_{H^\sigma} \leq M, \quad j = 1, 2, 3, \dots,$$

for some $\sigma \in \mathbb{R}$, where $\|g\|_{H^\sigma} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^\sigma |\widehat{g}(\xi)|^2 \, d\xi\right)^{1/2}$ denotes the H^σ Sobolev norm of g. Is it necessarily true that $\|f\|_{H^\sigma} < \infty$? Give a proof or counterexample.

6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.

a) Prove that if $1 \leq p \leq q \leq \infty$ are fixed then there is a constant A such that

$$\|f * \varphi\|_{L^q} \leq A \|f\|_{L^p}, \quad \text{for all} \quad f \in L^p.$$

If you use Young's (convolution) inequality, you should prove it.

b) Show by example that such a general inequality cannot hold for $p > q$.

7. Suppose that

$$f : [0, 1] \times [0, 1] \to \mathbb{R}$$

is continuous and has the property that for each x the map $t \to f(x, t)$ is differentiable and that $\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x)$ for some measurable function statisfying $\int_0^1 g(x) dx < \infty$. Carefully prove that $F(t) = \int_0^1 f(x, t) \, dx$ satisfies

$$F'(t) = \int_0^1 \frac{\partial f}{\partial t}(x, t) \, dx.$$

8. Let E be a measurable subset of the line.

a) Let $\chi_E : \mathbb{R} \to \mathbb{R}$ be the characteristic function of E (i.e. $\chi_E(x) = 1$ when $x \in E$ and $\chi_E(x) = 0$ when $x \notin E$). If E has finite Lebesgue measure, show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_{\mathbb{R}} \chi_E(y) \chi_E(y - x) dy$$

is continuous.

b) Suppose instead that E has positive Lebesgue measure $0 < |E| \leq \infty$. Using a), show that the set $E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.