ANALYSIS QUALIFYING EXAM
MAY 2011

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

1. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$. Give explicit formulas for the functions and give a proof for your answer. (An entire function is a holomorphic function on $\mathbb{C}$.)

2. Let $f : D \to \mathbb{C}$ be a holomorphic function with simple zeros at the points $1/3,\ 2/3,\ i/4$ and no other zeros. Determine the value of the integral

$$\int_{\{|z|=1/2\}} (z^2 - 1)\, e^z \, \frac{f'(z)}{f(z)}\, dz\,,$$

where the direction of integration is counterclockwise.

3. Let U be a bounded domain in $\mathbb{C}$, and let $f : U \to U$ such that f is holomorphic. Let $P \in U$ and suppose that $f(P) = P$. Prove that $|f'(P)| \leq 1$.

   Hint: Consider the sequence of iterates $f_n = f \circ f \circ \cdots \circ f$ (n times).

4. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is a harmonic function such that

$$u(z) \leq 10\, \log(|z| + 2)\,,$$

for all $z \in \mathbb{C}$. Prove that u is constant.

Part II. Real Analysis.

5. Let $f_n : [0,1] \to \mathbb{R}$, for $n = 1, 2, \ldots$, be a sequence of $\mathcal{C}^1$ functions such that $f_n(t) \le 5$ and $|f_n'(t)| \le 1$ for all n, t. Define the functions $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(t) = \max\{f_1(t), \ldots, f_n(t)\}$$

for $n = 1, 2, \ldots$. Prove that the sequence $\{g_n\}$ converges uniformly on $[0,1]$.

6. Let $f \in L^1(S^1)$ such that $\widehat{f} \in \ell^1(\mathbb{Z})$. Prove that $f \in \mathcal{C}(S^1)$ (continuous functions on the circle S^1).

7. Suppose that $f \in L^\infty([0,1])$.

   a) Prove that if $1 < p < \infty$ then $\|f\|_p \le \|f\|_\infty$.

   b) Show that $\|f\|_\infty \le \lim_{p\to\infty} \|f\|_p$ and therefore conclude that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

   Hint: Given $\varepsilon > 0$, consider $A_\varepsilon = \{x \in [0,1] : |f(x)| > \|f\|_\infty - \varepsilon\}$.

8. a) Let $f_j : \mathbb{R}^n \to \mathbb{R}$, for $j = 1, 2, \ldots$, be a sequence of L^2 functions. Suppose that there is a function $f \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f_j g \to \int_{\mathbb{R}^n} f g\,, \quad \forall g \in L^2(\mathbb{R}^n).$$

   Show that

$$\|f\|_2 \le \liminf_{j\to\infty} \|f_j\|_2.$$

   Also, give an example showing that strict inequality can occur.

   b) Suppose also that $\|f_j\|_2 \to \|f\|_2$. Show that in this case $\|f_j - f\|_2 \to 0$ as $j \to \infty$.