Analysis Qualifying Exam, May 9, 2012

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours

Part I. Complex Analysis

1. Use residues to calculate the integral $\int_0^\infty \frac{1}{(1+x^2)^2} dx$.

2. Suppose f is holomorphic on the open unit disc $D(0,1)$ and continuous on $\overline{D(0,1)}$. Assume $|f(\xi)| < 1$ for $\xi \in \partial D(0,1)$. Show that there exists an unique point $a \in D(0,1)$ such that $f(a) = a$.

3. Suppose f is holomorphic on $U := D(0,1) \setminus \{0\}$. Assume that the real part $\mathrm{Re}(f)$ is bounded from below on U. Prove that $z = 0$ is a removable singularity.

4. Let $U = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \leq \frac{\pi}{2}\}$and f be an entire function satisfying $f(U) \subset U$, $f(-1) = 0$, $f(0) = 1$. Prove that $f(z) = z + 1$.

Part II. Real Analysis

5. Justify or give a counterexample to the following assertions:
a. If $\{f_i\}$ is a sequence in $L^2([0,1])$ converging weakly to f in $L^2([0,1])$, then f_i^2 converges weakly to f^2 in $L^1([0,1])$.
b. If $\{f_i\}$ is a sequence in $L^2([0,1])$ converging strongly to f in $L^2([0,1])$, then f_i^2 converges strongly to f^2 in $L^1([0,1])$.

6. Let $\{g_k\}_{k=1}^\infty$ be a sequence in $L^1(\mathbb{R}^n)$ with $\sum ||g_k||_{L^1(\mathbb{R}^n)} < \infty$.
a. Show that $\sum_{k=1}^\infty g_k$ converges a.e. to a function $g \in L^1(\mathbb{R}^n)$.
b. Show that $\lim_{N\to\infty} ||g - \sum_{k=1}^N g_k||_{L^1(\mathbb{R}^n)} = 0$.

7. Let $f \in L^1(\mathbb{R})$ and set $h(x) = \int_{[x,\,x+1]} f(t)\ dt$.
a. Show that $h(x)$ is absolutely continuous.
b. Show that $\lim_{x\to\infty} h(x) = 0$.

8. Let $f \in L^1(\mathbb{R})$. Define its Fourier transform $\hat{f}(\xi) = \int f(x)e^{-2\pi i x\cdot\xi}\ dx$. Show that $\hat{f}(\xi) \in C_0(\mathbb{R})$, that is the Fourier transform is continuous and vanishes at infinity. *You may not quote the Riemann-Lebesgue lemma without sketching a proof.*