Qualifying Exam - Analysis

May, 2013

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Problem 1 Let $U \subset \mathbb{C}$ be an open set and let f be a continuous function on U. If f^2 is holomorphic on U, prove that f is holomorphic on U.

Problem 2 Prove that there is only one solution in the unit disc $\{z : |z| < 1\}$ and there are three solutions on the annulus $\{z : 1 < |z| < 2\}$ (counting multiplicities) for the equation $z^4 - 6z + 3 = 0$.

Problem 3 Let f be a holomorphic function on the unit disc $\{z : |z| < 1\}$ satisfying $f(0) = 0$ and $Ref(z) \leq A$ for some positive number $A > 0$. Prove:

$$|f(z)| \leq \frac{2A|z|}{1 - |z|}.$$

Problem 4 Calculate the following integral:

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{4 + x^2} dx.$$

Problem 5 Suppose that E and F are Lebesgue measurable sets of $\mathbb{R}$, and their Lebesgue measures $m(E) > 0$, $m(F) > 0$. Prove that

$$E + F = \{x + y : x \in E, y \in F\}$$

contains a nonempty open interval.

Problem 6(a) Prove the Riemann-Lebesgue Lemma: if $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of f,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \to 0, \text{ as } |\xi| \to \infty.$$

(b) Use part (a) to justify whether there exists a function $h \in L^1(\mathbb{R}^d)$ such that

$$f * h = f \text{ for all } f \in L^1(\mathbb{R}^d).$$

Here $f * h$ is the convolution of f and h defined by

$$(f * h)(x) = \int_{\mathbb{R}^d} f(x - y) h(y) dx.$$

Problem 7 If the sequence of Lebesgue measurable functions $\{f_n\}_{n=1}^\infty$ on $\mathbb{R}^d$ satisfying that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n(x)|^2 dx = 0,$$

show that there exists a subsequence of functions $\{f_{n_j}\}_{j=1}^\infty$ such that

$$f_{n_j}(x) \to 0 \text{ a.e. } x.$$

Problem 8 Recall that the inner product on $L^2(\mathbb{R}^d)$ is given by

$$(f,g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx, \text{ for } f,g \in L^2(\mathbb{R}^d),$$

which induces the L^2-norm

$$\|f\|_{L^2} = (f,f)^{1/2}.$$

(a) If the sequence of functions $\{f_n\}_{n=1}^\infty$ in $L^2(\mathbb{R}^d)$ satisfy that $\|f_n\|_{L^2} = 1$, show that there exists a subsequence of functions $\{f_{n_j}\}_{j=1}^\infty$ such that f_{n_j} converges weakly to some function f in $L^2(\mathbb{R}^d)$, i.e.,

$$(f_{n_j}, g) \to (f,g) \text{ for all } g \in L^2(\mathbb{R}^d).$$

(b) If $f_n \to f$ weakly in $L^2(\mathbb{R}^d)$ and $\|f_n\|_{L^2} \to \|f\|_{L^2}$ as $n \to \infty$, show that $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.