QUALIFYING EXAM - ANALYSIS - SPRING 2014

1. Prove the following statement without using Ergoroff's Theorem: Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E, $f_k \to f$ a.e. on E and there exists $g \in L^1(E)$ such that $|f_k| \leq g$ for all k. Given $\epsilon > 0$, there exists a closed set A_ϵ such that $m(E \backslash A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

2. Let $f \in L^1(\mathbb{R})$ and define $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that
$$\int_{\mathbb{R}} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove the following statement: There exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'| \leq M$.

4. (a) Prove that the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by setting $T[f](x) = xf(x)$ is continuous and symmetric (self-adjoint).

(b) Prove that T is not compact.

5. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \to D$ be a holomorphic function. Prove
$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}, \quad \forall z \in D.$$

6. For $t \in \mathbb{R}$, compute
$$\lim_{A \to \infty} \int_{-A}^{A} \frac{\sin x}{x} e^{ixt} dx.$$

7. Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$. Prove that if $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0.

8. Prove that if f is an entire function and $|f(z)| \leq a + b|z|^k$ for all $z \in \mathbb{C}$ where a, b and k are positive real numbers, then f is a polynomial.