Qualifying Exam - Analysis - Spring 2015

Justify your answers to all problems.

1. Assume $f, f_j \subset L^2(\mathbb{R}^n)$ for $j = 1, 2, \ldots, f_j \to f$ a.e. and $\int f_j^2 dx \to \int f^2 dx$. Prove $\int |f_j - f|^2 dx \to 0$.

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-negative, $C^\infty$ function with compact support such that

\[ \int_{\mathbb{R}} \varphi(x) dx = 1. \]

Define

\[ \varphi_{\sigma}(x) = \sigma^{-1} \varphi\left(\frac{x}{\sigma}\right) \quad \text{and} \quad u_{\sigma}(x) = \int \varphi_{\sigma}(x - y) u(y) dy. \]

For $u \in L^2(\mathbb{R})$, prove

\[ \int_{\mathbb{R}} |u_{\sigma}(x)|^2 dx \leq \int_{\mathbb{R}} |u(x)|^2 dx. \]

3. Assume $f : [0, 1] \to \mathbb{R}$ is uniformly continuous, increasing and convex. Prove $f$ is differentiable almost everywhere and

\[ f(1) - f(0) = \int_0^1 f'(x) dx. \]

4. Assume $f : [0, 1] \to \mathbb{R}$ is a measurable function such that $fg \in L^1([0, 1])$ for all $g \in L^2([0, 1])$. Prove $f \in L^2([0, 1])$.

5. Let $U \subset \mathbb{C}$ be an open set. Assume $f, g : U \to \mathbb{C}$ are holomorphic function such that $fg$ is holomorphic. Prove either $f$ is constant or $g$ is identically equal to 0.

6. Assume $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire function. Prove $f(\mathbb{C})$ is dense in $\mathbb{C}$.

7. Prove that $z^5 + 3z^3 + 7$ has all its zeros in the disk $D(0, 2) = \{z \in \mathbb{C} : |z| < 2\}$.

8. Let $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Assume $r > 1$ and $f : \overline{D(0, r) \setminus D(0, 1)} \to \mathbb{C}$ is a continuous function, holomorphic on $D(0, r) \setminus \overline{D(0, 1)}$ that satisfies

\[ \max_{\partial D(0, 1)} |f(z)| = 1 \quad \text{and} \quad \max_{\partial D(0, r)} |f(z)| = R. \]

Prove $\log |f(z)| \leq \frac{\log R}{\log r} \log |z|$. 

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