Qualifying Exam - Analysis - Spring 2015

Justify your answers to all problems.

1. Assume $f, f_j \subset L^2(\mathbb{R}^n)$ for $j = 1, 2, \ldots,$ $f_j \to f$ a.e. and $\int f_j^2 dx \to \int f^2 dx$. Prove $\int |f_j - f|^2 dx \to 0$.

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-negative, C^∞ function with compact support such that

$$\int_{\mathbb{R}} \varphi(x)dx = 1.$$

Define

$$\varphi_\sigma(x) = \sigma^{-1}\varphi(\frac{x}{\sigma}) \text{ and } u_\sigma(x) = \int \varphi_\sigma(x-y)u(y)dy.$$

For $u \in L^2(\mathbb{R})$, prove

$$\int_{\mathbb{R}} |u_\sigma(x)|^2 dx \leq \int_{\mathbb{R}} |u(x)|^2 dx.$$

3. Assume $f : [0,1] \to \mathbb{R}$ is uniformly continuous, increasing and convex. Prove f is differentiable almost everywhere and

$$f(1) - f(0) = \int_0^1 f'(x)dx.$$

4. Assume $f : [0,1] \to \mathbb{R}$ is a measurable function such that $fg \in L^1([0,1])$ for all $g \in L^2([0,1])$. Prove $f \in L^2([0,1])$.

5. Let $U \subset \mathbb{C}$ be an open set. Assume $f, g : U \to \mathbb{C}$ are holomorphic function such that $\bar{f}g$ is holomorphic. Prove either f is constant or g is identically equal to 0.

6. Assume $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire function. Prove $f(\mathbb{C})$ is dense in $\mathbb{C}$.

7. Prove that $z^5 + 3z^3 + 7$ has all its zeros in the disk $D(0,2) = \{z \in \mathbb{C} : |z| < 2\}$.

8. Let $D(0,r) = \{z \in C : |z| < r\}$. Assume $r > 1$ and $f : \overline{D(0,r)} \backslash D(0,1) \to \mathbb{C}$ is a continuous function, holomorphic on $D(0,r) \backslash \overline{D(0,1)}$ that satisfies

$$\max_{\partial D(0,1)} |f(z)| = 1 \text{ and } \max_{\partial D(0,r)} |f(z)| = R.$$

Prove $\log |f(z)| \leq \dfrac{\log R}{\log r} \log |z|$.