QUALIFYING EXAM SPRING 2016 - ANALYSIS

1. Prove the absolute continuity of the Lebesgue integral; in other words, prove that if f is integrable on $\mathbb{R}^d$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f| < \epsilon \text{ whenever } m(E) < \delta.$$

2. Prove that the Hardy-Littlewood maximal function f^* for an integrable function f satisfies

$$m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\} \leq \frac{3^d}{\alpha} ||f||_{L^1(\mathbb{R}^d)}$$

where $\alpha > 0$. Recall that

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken over all balls containing the point x. You may assume the Vitali 3-times Covering Lemma. *State it clearly if you use it.*

3. Let $f : [0,1] \to [0,1]$ be a continuous function and $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $\phi(0) = 0$. Prove

$$\int_0^1 \phi \circ f \, dx = \int m(\{x \in [0,1] : f(x) > t\}) \phi'(t) dt$$

4. Let $U \subset \mathbb{C}$ be an open set and

$$A^2(U) = \{f \text{ holomorphic on } U : \int_U |f(z)|^2 dxdy < \infty\}.$$

Define

$$< f, g >= \int_U f(z)\overline{g(z)} dxdy, \quad \forall f, g \in A^2(U).$$

Prove that $A^2(U)$ is a Hilbert space when equipped with this inner product.

5. Let $f : D \to D$ be a holomorphic function where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. Prove that if f has at least 2 fixed points then f is the identity map. (Note: A point a is said to be a fixed point of f if $f(a) = a$.)

6. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function, not identically equal to 0 and the let $\mathcal{Z} = \{z \in \mathbb{C} : f(z) = 0\}$. Prove that if $\mathcal{Z}$ is unbounded, then f has an essential singularity at ∞.

7. Determine the number of zeroes of the polynomial

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| \leq 2$.