Justify your answers to all problems.

1. Let \( f : \mathbb{R} \to [0, \infty) \) be a measurable function and \( \varphi : [0, \infty) \to [0, \infty) \) be a monotonic, absolutely continuous function on \([0, T]\) for every \( T < \infty \). Assume \( \varphi(0) = 0 \). Prove
\[
\int_{\mathbb{R}} \varphi \circ f \, dx = \int_0^\infty m(\{x : f(x) > t\}) \varphi'(t) \, dt.
\]

2. Let \( H \) be a Hilbert space equipped with an inner product \((\cdot, \cdot)\) and a norm \( ||\cdot|| = (\cdot, \cdot)^{\frac{1}{2}} \). Recall the following: A sequence \( \{f_k\} \subset H \) is said converge to \( f \in H \) if \( ||f_k - f|| \to 0 \). A sequence \( \{f_k\} \subset H \) is said converge weakly to \( f \in H \) if \( (f_k, g) \to (f, g) \) for any \( g \in H \). Prove the following statements:
   (a) \( \{f_k\} \) converges to \( f \) if and only if \( ||f_k|| \to ||f|| \) and \( \{f_k\} \) converges weakly to \( f \).
   (b) If \( H \) is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.
   (c) If a sequence \( \{f_k\} \) converges weakly to \( f \), then there exists a subsequence \( \{f_{k_n}\} \) such that
   \[
f_{k_1} + \cdots + f_{k_n}
   \]
   converges to \( f \). (You may use the fact that a weakly convergent sequence is a bounded sequence.)

3. Let \( \{E_k\} \) be a sequence of (Lebesgue) measurable sets in \( \mathbb{R}^k \) such that
\[
\sum_{k=1}^\infty m(E_k) < \infty.
\]
Prove that almost every \( x \in \mathbb{R}^k \) lie in at most finitely many sets \( E_k \).

4. Let \( U \subset \mathbb{C} \) be an open set, \( D = \{z \in \mathbb{C} : |z| < 1\} \) and \( \mathcal{F} \) be the set of all holomorphic functions \( f : U \to D \). Given \( z_0 \in U \), show that there exists \( f_0 \in \mathcal{F} \) such that
\[
|f_0''(z_0)| = \sup_{f \in \mathcal{F}} |f''(z_0)|.
\]

5. Describe all holomorphic functions on \( \mathbb{C} \setminus \{0\} \) with the property that
\[
|f(z)| \leq |z|^2 + \frac{1}{|z|^\frac{1}{2}}, \quad \forall z \in \mathbb{C} \setminus \{0\}.
\]

6. Let \( f : U \to \mathbb{C} \) be a non-constant holomorphic function where \( U \subset \mathbb{C} \) is an open set containing the closure \( \overline{D} \) of the unit disk \( D = \{z \in \mathbb{C} : |z| < 1\} \). If \( |f(z)| = 1 \) for all \( z \in \partial D \), then prove that \( D \subset f(\overline{D}) \).