Qualifying Exam - Analysis - Spring 2017

Justify your answers to all problems.

1. Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function and $\varphi : [0, \infty) \to [0, \infty)$ be a monotonic, absolutely continuous function on $[0, T]$ for every $T < \infty$. Assume $\varphi(0) = 0$. Prove

$$\int_{\mathbb{R}} \varphi \circ f \; dx = \int_0^\infty m(\{x : f(x) > t\})\varphi'(t) \; dt.$$

2. Let $\mathcal{H}$ be a Hilbert space equipped with an inner product $(\cdot, \cdot)$ and a norm $||\cdot|| = (\cdot, \cdot)^{\frac{1}{2}}$. Recall the following: A sequence $\{f_k\} \subset \mathcal{H}$ is said *converge* to $f \in \mathcal{H}$ if $||f_k - f|| \to 0$. A sequence $\{f_k\} \subset \mathcal{H}$ is said *converge weakly* to $f \in \mathcal{H}$ if $(f_k, g) \to (f, g)$ for any $g \in \mathcal{H}$. Prove the following statements:

(a) $\{f_k\}$ converges to f if and only if $||f_k|| \to ||f||$ and $\{f_k\}$ converges weakly to f.

(b) If $\mathcal{H}$ is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.

(c) If a sequence $\{f_k\}$ converges weakly to f, then there exists a subsequence $\{f_{k_n}\}$ such that

$$\frac{f_{k_1} + \cdots + f_{k_n}}{n}$$

converges to f. (You may use the fact that a weakly convergent sequence is a bounded sequence.)

3. Let $\{E_k\}$ be a sequence of (Lebesgue) measurable sets in $\mathbb{R}^k$ such that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Prove that almost every $x \in \mathbb{R}^k$ lie in at most finitely many sets E_k.

4. Let $U \subset \mathbb{C}$ be an open set, $D = \{z \in C : |z| < 1\}$ and $\mathcal{F}$ be the set of all holomorphic functions $f : U \to D$. Given $z_0 \in U$, show that there exists $f_0 \in \mathcal{F}$ such that

$$|f_0''(z_0)| = \sup_{f \in \mathcal{F}} |f''(z_0)|.$$

5. Describe all holomorphic functions on $\mathbb{C}\backslash\{0\}$ with the property that

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{\frac{1}{2}}}, \quad \forall z \in \mathbb{C}\backslash\{0\}.$$

6. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function where $U \subset \mathbb{C}$ is an open set containing the closure $\overline{D}$ of the unit disk $D = \{z \in C : |z| < 1\}$. If $|f(z)| = 1$ for all $z \in \partial D$, then prove that $D \subset f(\overline{D})$.