# COMPLEX ANALYSIS CORE QUALIFYING EXAM, FALL 2001

**Instructions:** Attempt **FOUR** of the following problems. Each is worth 25 points. Please label clearly which four of the five problems you want graded. Show all your work.

**Notation:** $\mathbb{C}$ denotes the complex numbers. For $z \in \mathbb{C}$, $\text{Re}(z)$ denotes the real part of $z$. For each $r \geq 0$, $D_r(0) = \{z \in \mathbb{C} : |z| < r\}$.

**Problem 1.** A meromorphic function on $\mathbb{C} \cup \{\infty\}$ is a meromorphic function $f(z)$ on $\mathbb{C}$ such that $g(z) = f(1/z)$ is also meromorphic. Show that a meromorphic function on $\mathbb{C} \cup \{\infty\}$ must be rational, i.e. one can express it as the quotient of two polynomials.

**Problem 2.** Fix a real number $\alpha > 1$. Show that the equation $z - \alpha = e^{-z}$ has precisely one solution in the half plane $\text{Re}(z) > 0$ and that this solution must be real.

**Problem 3.** Compute: $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$.

**Problem 4.** Suppose that $f : D_1(0) \to \mathbb{C}$ is a one-to-one holomorphic function with $\Omega = f\left(D_1(0)\right)$. Let $g : D_1(0) \to \Omega$ be another holomorphic function with $g(0) = f(0)$. Show that for each $0 \leq r < 1$, $g\left(D_r(0)\right) \subset f\left(D_r(0)\right)$.

**Problem 5.** Use the result in Problem 4 to prove the following: If $g$ is a holomorphic function on $D_1(0)$ with $g(0) = 0$ and $|\text{Re}(g(z))| < 1$ for all $z \in D_1(0)$, then

$$|g(z)| \leq \frac{2}{\pi} \log\left\{\frac{1+|z|}{1-|z|}\right\}$$

for all $z \in D_1(0)$.