Complex Analysis Core Qualifying Exam, Fall 2002

Do 5 of the 6 problems. Indicate clearly which 5 you want graded; if it is not clear, we will grade #1–5. Each problem counts for 20 points. In the case where there are two parts, the score is subdivided as indicated. Note: for the purposes of the exam, *holomorphic* is the same as *complex analytic*.

1. (a) (5 points) Give a counterexample to the assertion: If f is holomorphic on the annulus $\{z : 1 < |z| < 3\}$, then f extends holomorphically to the disc $\{z : |z| < 3\}$.

   (b) (15 points) Determine whether the following is true: If f is holomorphic on the annulus $\{z : 1 < |z| < 3\}$, then f extends meromorphically to the disc $\{z : |z| < 3\}$.

2. (a) (15 points) Show that there is no one-to-one holomorphic mapping of the open annulus $\{z : 1 < |z| < 2\}$ onto the punctured unit disc $\{z : 0 < |z| < 1\}$. (HINT: consider the inverse mapping)

   (b) (5 points) Give an example of a one-to-one C^∞ mapping of the open annulus $\{z : 1 < |z| < 2\}$ onto the punctured unit disc $\{z : 0 < |z| < 1\}$.

3. (20 points) Determine all entire functions f for which $|f(z)| \leq |z|^2$ for all $z \in \mathbb{C}$.

4. (20 points) Let D denote the unit disc $\{z : |z| < 1\}$. Determine a holomorphic mapping f of D *onto* itself for which $f(\frac{1}{2}) = -\frac{1}{\pi}$.

5. Let $P(z) = z^7 + z^3 + \frac{1}{16}$.

   (a) (5 points) Show that P has no multiple zeros.

   (b) (15 points) Determine the number of zeros of P that lie in the closed disc $|z| \leq \frac{1}{2}$.

6. (20 points) Evaluate the integral:

$$\int_0^\infty \frac{u^2\, du}{u^6 + 1}$$