REAL ANALYSIS QUALIFYING EXAM, SPRING 2001

Instructions: Attempt to do all of the problems. Each is worth 20 points. All the measures involved are Lebesgue measure.

1.) Suppose that $\phi \in C_0^\infty(\mathbb{R}^n)$ has $\int \phi dx = 1$. If $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, prove that if $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$ then $f * \phi_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$. Prove that this is not true for $p = \infty$.

2.) Suppose that $f \in L^1(\mathbb{R}^n)$. Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if A is measurable with measure $< \delta$ then

$$|\int_A f dx| < \varepsilon.$$

3.) Recall that $f : [0,1] \to \mathbb{R}$ is lower semicontinuous if $\liminf_{x \to x_0} f(x) \ge f(x_0)$ for every $x_0 \in [0,1]$. Prove that if f is a nonnegative lower semicontinuous function then one always has $S_+(f,P) \to \int_0^1 f(x)dx$ as $|P| \to 0$ if $S_+(f,P)$ is the lower Riemann sum associated with a partition P of $[0,1]$ and $|P|$ is the smallest interval of the partition. Here $\int_0^1 f(x)dx$ is the Lebesgue integral of f.

Here, if $0 = t_0 < t_1 < \cdots < t_n = 1$, is the partition P, then

$$S_+(f,P) = \sum_{j=1}^{n} \inf_{x \in [t_{j-1}, t_j)} f(x)(t_j - t_{j-1}).$$

Hint: To prove $S_+(f,P) \to \int_0^1 f(x)dx$ as $|P| \to 0$, it suffices to show that $S_+(f,P_n) \to \int_0^1 f(x)dx$ if P_n is a nested sequence of partitions whose lengths goes to zero.

4.) For which values of α and β does the following inequality hold?

$$\|f\|_2 \le \|f\|_{4/3}^\alpha \|f\|_4^\beta.$$

Prove your assertion.

5.) Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$ define

$$Tf(x) = \int_0^1 K(x,y)f(y)dy.$$

Prove that $Tf \in C([0,1])$. Moreover, prove that $\Omega = \{Tf : \|f\|_{sup} \le 1\}$ is precompact in $C([0,1])$. Here, we are using the sup-norm $\|\cdot\|_{sup}$ on $C([0,1])$ and Ω being precompact means that every sequence in Ω must have a subsequence that converges with respect to this norm to an element of $C([0,1])$.