Complex Analysis Core Qualifying Exam
Spring 2002
Instruction: Answer any FOUR questions

1. Let f be an entire function such that the image of f does not intersect $\{z \in \mathbb{R} : z \geq 5\}$. Prove that f is a constant.

2. Evaluate the integral
$$\int_0^{2\pi} \frac{dx}{a^2 + \cos^2 x}.$$
Where $a > 1$.

3. Classify all simply connected regions in the extended complex plane up to biholomorphic equivalence. i.e, give a list of simply connected region, prove that every simply connected region in the extended complex plane is biholomorphic equivalent to a member in your list. Prove also that no two members in your list are biholomorphic equivalent.

4. Let f be a holomorphic function which maps the unit disk into the unit disc. Show that
$$|f(z) + f(-z)| \leq 2|z|^2$$
for all z in the unit disc, and if the equality holds for some z, then,
$$f(z) = e^{i\theta} z^2$$
for some real θ.

5. Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series expansion of $\frac{1}{\sin z}$ on the annulus $\{z \in \mathbb{C} : \pi < |z| < 2\pi\}$. Evaluate the coefficients a_n for $n < 0$.

6. Show that a Möbius transformation maps a straight line or circle onto a straight line or circle.