Please attempt all the problems and show all your work. In the following, “holomorphic” is synonymous with “analytic.” Also, $\Delta$ will denote the open unit disk in $\mathbb{C}$.

(1) (a) Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic with a pole at infinity. Show that $f$ must be a rational function.

(b) Use the above to prove the following: if $f : \Delta \to \mathbb{C}$ is holomorphic with a continuous extension to the boundary of $\Delta$ such that $|f(z)| = 1$ for all $|z| = 1$, then $f(z)$ is the restriction of a rational function.

(2) Let $f : \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$ and $|f'(0)| = M$. If $0 \neq w \in \Delta$ is any other zero of $f(z)$, show that:

$$\frac{M}{1 + M} \leq |w|.$$ 

(3) Let $C$ be the closed curve defined by two pieces: the first piece is given by the set of all $z$ satisfying $|z - 1| = 3$ and $\text{Re}(z - 1) \geq 0$. The second piece is the straight line segment from $1 + 3i$ to $1 - 3i$. Orient $C$ in the counterclockwise direction, and let $\Omega$ be the region enclosed by $C$. Suppose $f$ is holomorphic in a neighborhood of $\Omega$ with no zeros on $C$. Suppose also that:

$$\frac{1}{2\pi i} \int_C \frac{zf'(z)}{f(z)} \, dz = 3 \quad \text{and} \quad \frac{1}{2\pi i} \int_C \frac{z^2f'(z)}{f(z)} \, dz = \frac{5}{2}.$$ 

Determine all the zeros of $f$ in $\Omega$ explicitly.

(4) (a) State Rouché’s Theorem.

(b) Let $\varphi : \Omega \to \mathbb{C}$ be holomorphic on an open convex set $\Omega$. Show that for $z, w \in \Omega$

$$|\varphi(z) - \varphi(w)| \leq \max_{\xi \in L} |\varphi'(|\xi)||z - w|,$$

where $L$ is the straight line segment from $z$ to $w$.

(c) Use the above to prove the following: suppose

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

where

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$ 

Show that $f(z)$ is a 1-1 holomorphic function on $\Delta$. 