SPRING 2003 COMPLEX ANALYSIS QUALIFYING EXAM

Please attempt all the problems and show all your work. In the following, "holomorphic" is synonymous with "analytic." Also, Δ will denote the open unit disk in $\mathbb{C}$.

(1) (a) Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic with a pole at infinity. Show that f must be a rational function.

(b) Use the above to prove the following: if $f : \Delta \to \mathbb{C}$ is holomorphic with a continuous extension to the boundary of Δ such that $|f(z)| = 1$ for all $|z| = 1$, then $f(z)$ is the restriction of a rational function.

(2) Let $f : \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$ and $|f'(0)| = M$. If $0 \neq w \in \Delta$ is any other zero of $f(z)$, show that:

$$\frac{M}{1+M} \leq |w| \; .$$

(3) Let C be the closed curve defined by two pieces: the first piece is given by the set of all z satisfying $|z-1| = 3$ and $\mathrm{Re}(z-1) \geq 0$. The second piece is the straight line segment from $1+3i$ to $1-3i$. Orient C in the counterclockwise direction, and let Ω be the region enclosed by C. Suppose f is holomorphic in a neighborhood of $\overline{\Omega}$ with no zeros on C. Suppose also that:

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz = 3 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_C \frac{z^2 f'(z)}{f(z)}dz = \frac{5}{2} \; .$$

Determine all the zeros of f in Ω explicitly.

(4) (a) State Rouché's Theorem.

(b) Let $\varphi : \Omega \to \mathbb{C}$ be holomorphic on an open convex set Ω. Show that for $z, w \in \Omega$

$$|\varphi(z) - \varphi(w)| \leq \max_{\xi \in L} |\varphi'(\xi)||z-w| \; ,$$

where L is the straight line segment from z to w.

(c) Use the above to prove the following: suppose

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

where

$$\sum_{n=2}^{\infty} n|a_n| \leq 1 \; .$$

Show that $f(z)$ is a 1-1 holomorphic function on Δ.