Real Analysis Qualifying Exam, Fall 2001

Instructions: Attempt to do all problems. Each is worth 20 points. All the measures involved are Lebesgue measure.

1.) Let \( f \) be a continuous function on \([0, \infty)\) such that \( \lim_{x \to \infty} f(x) \) exists (finitely). Prove that \( f \) is uniformly continuous.

2.) Let \( f \) and \( g \) be continuous real valued functions on \( \mathbb{R} \) such that \( \lim_{|x| \to \infty} f(x) = 0 \) and \( \int_{-\infty}^{\infty} |g(x)| \, dx < \infty \). Define the function \( h \) on \( \mathbb{R} \) by

\[
h(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy.
\]

Prove that \( \lim_{|x| \to \infty} h(x) = 0 \).

3.) Let \( \{f_n\} \) be a sequence of real valued functions in \( L^{4/3}(0,1) \) such that \( f_n \to 0 \) in measure as \( n \to \infty \) and \( \int_0^1 |f_n(x)|^{4/3} \, dx \leq 1 \). Show that \( \int_0^1 |f_n(x)| \, dx \to 0 \) as \( n \to \infty \).

4.) Let \( f \in L^1([0,1]) \). For \( k \in \mathbb{N} \), let \( f_k \) be the step function defined on \([0,1]\) by

\[
f_k(x) = k \int_{j/k}^{(j+1)/k} f(t) \, dt, \quad \text{for} \quad \frac{j}{k} \leq x < \frac{j+1}{k}.
\]

Show that \( f_k \) tends to \( f \) in \( L^1 \) norm as \( k \) tends to \( +\infty \).

Hint: Treat first the case where \( f \) is continuous, and use approximation.

5.) Let \( 1 \leq p < q < \infty \). Which of the following statements (i)-(vi) are true, and which are false? Justify all the negative answers by a counterexample, but you do not have to justify the positive answers.

(i) \( L^p(\mathbb{R}) \subset L^q(\mathbb{R}) \).
(ii) \( L^q(\mathbb{R}) \subset L^p(\mathbb{R}) \).
(iii) \( L^p([0, 1]) \subset L^q([0, 1]) \).
(iv) \( L^q([0, 1]) \subset L^p([0, 1]) \).
(v) \( \ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z}) \).
(vi) \( \ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \).

Justify your answer to the following question:
(vii) For which \( s \geq 1 \) is \( L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R}) \)?