Real Analysis Qualifying Exam, Fall 2001

Instructions: Attempt to do all problems. Each is worth 20 points. All the measures involved are Lebesgue measure.

1.) Let f be a continuous function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x)$ exists (finitely). Prove that f is uniformly continuous.

2.) Let f and g be continuous real valued functions on $\mathbb{R}$ such that $\lim_{|x|\to\infty} f(x) = 0$ and $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. Define the function h on $\mathbb{R}$ by

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy.$$

Prove that $\lim_{|x|\to\infty} h(x) = 0$.

3.) Let $\{f_n\}$ be a sequence of real valued functions in $L^{4/3}(0,1)$ such that $f_n \to 0$ in measure as $n \to \infty$ and $\int_0^1 |f_n(x)|^{4/3}dx \le 1$. Show that $\int_0^1 |f_n(x)|dx \to 0$ as $n \to \infty$.

4.) Let $f \in L^1([0,1])$. For $k \in \mathbb{N}$, let f_k be the step function defined on $[0,1]$ by

$$f_k(x) = k\int_{j/k}^{(j+1)/k} f(t)\, dt, \quad \text{for } \frac{j}{k} \le x < \frac{j+1}{k}.$$

Show that f_k tends to f in L^1 norm as k tends to $+\infty$.

Hint: Treat first the case where f is continuous, and use approximation.

5.) Let $1 \le p < q < \infty$. Which of the following statements (i)-(vi) are true, and which are false? Justify all the negative answers by a counterexample, but you do not have to justify the positive answers.

(i) $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$.

(ii) $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$.

(iii) $L^p([0,1]) \subset L^q([0,1])$.

(iv) $L^q([0,1]) \subset L^p([0,1])$.

(v) $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$.

(vi) $\ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z})$.

Justify your answer to the following question:

(vii) For which $s \geq 1$ is $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R})$?