1. Let $\psi(x) = x$ on $[0, \frac{1}{2}]$, $\psi(x) = 1 - x$ on $[\frac{1}{2}, 1]$ and extended periodically of period 1. Define $f(x) = \sum_{n=0}^{\infty} 2^{-n} \psi(8^n x)$. 
   i. Show that $f(x)$ is continuous everywhere.
   ii. Show that $f(x)$ is differentiable nowhere.
   Hint: Consider the difference quotients
   \[ \Delta_h f(x) \equiv \frac{f(x + h) - f(x)}{h} \]
   where $h = \pm 8^{-k}$ and the sign is chosen so that $x$ and $x + h$ lie on the same linear segment of the graph of $\psi(8^{k-1}x)$. Then
   a. $\Delta_h f(x) = \sum_{n=0}^{k-1} 2^{-n} \Delta_h \psi(8^n x)$
   b. $|\Delta_h f(x)| \geq 4^{k-1} - \sum_{n=0}^{k-2} 4^n$

2. Let $f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots$ on a set $A$, where the functions $f_n$ are integrable and $\int_A f_n(x) \, dx \leq M$ for some constant $M$. Show that the limit
   \[ f(x) = \lim_{n \to \infty} f_n(x) \]
   exists and is finite almost everywhere on $A$ and that
   \[ \lim_{n \to \infty} \int_A f_n(x) \, dx = \int_A f(x) \, dx . \]

3. i. Define equicontinuity and state the Arzela-Ascoli theorem.
   ii. Let $\mathcal{F}$ be the family of real valued functions on $[0, 1]$ satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2 \, dx \leq 1$. Show that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly.

4. Let $K$ be a closed convex subset of a Hilbert space $H$. Show that for each $x \in H$, there is a unique $y \in K$ such that
   \[ ||x - y|| = \inf_{z \in K} ||x - z|| \]

5. i. Find the sum of the series $\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$ on $(0, 2\pi)$.
   ii. Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.