Real Analysis Qualifying Exam, Fall 2002

Instructions: You have 2 hours to do all problems as completely as possible.

1. Let $\psi(x) = x$ on $[0, \frac{1}{2}]$, $\psi(x) = 1 - x$ on $[\frac{1}{2}, 1]$ and extended periodically of period 1. Define $f(x) = \sum_{n=0}^{\infty} 2^{-n}\psi(8^n x)$.
i. Show that $f(x)$ is continuous everywhere.
ii. Show that $f(x)$ is differentiable nowhere.
Hint: Consider the difference quotients

$$\Delta_h f(x) \equiv \frac{f(x+h) - f(x)}{h}$$

where $h = \pm 8^{-k}$ and the sign is chosen so that x and $x + h$ lie on the same linear segment of the graph of $\psi(8^{k-1}x)$. Then
a. $\Delta_h f(x) = \sum_{n=0}^{k-1} 2^{-n} \Delta_h \psi(8^n x)$
b. $|\Delta_h f(x)| \geq 4^{k-1} - \sum_{n=0}^{k-2} 4^n$

2. Let $f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots$ on a set A, where the functions f_n are integrable and $\int_A f_n(x)\, dx \leq M$ for some constant M. Show that the limit

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists and is finite almost everywhere on A and that

$$\lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A f(x)\, dx \, .$$

3. i. Define equicontinuity and state the Arzela-Ascoli theorem.
ii. Let $\mathcal{F}$ be the family of real valued functions on $[0, 1]$ satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2\, dx \leq 1$ Show that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly.

4. Let K be a closed convex subset of a Hilbert space H. Show that for each $x \in H$, there is a unique $y \in K$ such that

$$||x - y|| = inf_{z\in K}||x - z||$$

5. i. Find the sum of the series $\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$ on $(0, 2\pi)$.
ii. Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$