Real Analysis Qualifying Exam

Spring 2002

Time: 2 hours

Instructions: Do five of the following 6 problems. (If you attempt all 6 problems, clearly indicate which problems you want graded.) Each problem is worth 20 points.

1. Let \( f : [0, 2] \to \mathbb{R} \) be a \( C^1 \) function such that \( f(x) \) and \( f'(x) \) vanish at \( x = 0 \) and at \( x = 2 \). Prove that for all \( \varepsilon > 0 \) there exists \( t_\varepsilon \in \mathbb{R}^+ \) such that

\[
\left| \int_0^2 f(x)e^{itx} \, dx \right| \leq \frac{\varepsilon}{t} \quad \text{for} \quad t \geq t_\varepsilon .
\]

2. Let \( \{c_n\} \) be a sequence of positive real numbers, and let \( f_n : \mathbb{R} \to \mathbb{R} \) be given by

\[
f_n(x) = \sin(x + c_n^2) + \frac{1}{c_n} \sin(c_nx) .
\]

Prove that the sequence \( \{f_n\} \) has a subsequence converging pointwise to a continuous function.

3. Let \( X \) denote the set of functions \( f : [0, 1] \to \mathbb{R} \) such that \( \|f\| < \infty \), where

\[
\|f\| := |f(0)| + \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/5}} : x \neq y \right\} .
\]

Prove that \( (X, \| \cdot \|) \) is a Banach space; i.e., show that \( X \) is a vector space, \( \| \cdot \| \) is a norm, and \( X \) is complete.

4. Suppose that \( f \in L^1(\mathbb{R}^n, m) \) satisfies

\[
\left| \int_E f \, dm \right| \leq m(E)
\]

for all Lebesgue measurable sets \( E \) (where \( m \) denotes Lebesgue measure on \( \mathbb{R}^n \)). Prove that \( |f| \leq 1 \) almost everywhere.

5. Let \( (X, \mathcal{M}, \mu) \) be a measure space, and let \( f \in L^1(\mu) \cap L^\infty(\mu) \). Prove that

\[
\lim_{p \to \infty} \|f\|_p = \|f\|_\infty .
\]

6. Let \( u \in \mathcal{D}'(\mathbb{R}) \) be given by

\[
(u, \varphi) = \lim_{\varepsilon \to 0^+} \left[ \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} \, dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x} \, dx \right] , \quad \forall \varphi \in \mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) .
\]

Show that the above limit exists and that \( u \) is the distribution derivative of the function \( f \in L^1_{\text{loc}}(\mathbb{R}) \) given by \( f(x) = \log |x| \).