Real Analysis Qualifying Exam **Spring 2002**

Time: 2 hours

Instructions: Do **five** of the following 6 problems. (If you attempt all 6 problems, clearly indicate which problems you want graded.) Each problem is worth 20 points.

1. Let $f : [0,2] \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $f(x)$ and $f'(x)$ vanish at $x = 0$ and at $x = 2$. Prove that for all $\varepsilon > 0$ there exists $t_\varepsilon \in \mathbb{R}^+$ such that

$$\left| \int_0^2 f(x) e^{itx}\, dx \right| \leq \frac{\varepsilon}{t} \qquad \text{for } \; t \geq t_\varepsilon \; .$$

2. Let $\{c_n\}$ be a sequence of positive real numbers, and let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = \sin(x + c_n^2) + \frac{1}{c_n} \sin(c_n x) \; .$$

Prove that the sequence $\{f_n\}$ has a subsequence converging pointwise to a continuous function.

3. Let X denote the set of functions $f : [0,1] \to \mathbb{R}$ such that $\|\!|f|\!\|< \infty$, where

$$\|\!|f|\!\| := |f(0)| + \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^{1/5}} : x \neq y \right\} \; .$$

Prove that $(X, \|\!|\cdot|\!\|)$ is a Banach space; i.e., show that X is a vector space, $\|\!|\cdot|\!\|$ is a norm, and X is complete.

4. Suppose that $f \in L^1(\mathbb{R}^n, m)$ satisfies

$$\left| \int_E f\, dm \right| \leq m(E)$$

for all Lebesgue measurable sets E (where m denotes Lebesgue measure on $\mathbb{R}^n$). Prove that $|f| \leq 1$ almost everywhere.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \in L^1(\mu) \cap L^\infty(\mu)$. Prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty \; .$$

6. Let $u \in \mathcal{D}'(\mathbb{R})$ be given by

$$(u, \varphi) = \lim_{\varepsilon \to 0^+} \left[\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\, dx \right] , \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}) = \mathcal{C}_c^\infty(\mathbb{R}) \; .$$

Show that the above limit exists and that u is the distribution derivative of the function $f \in L^1_{\text{loc}}(\mathbb{R})$ given by $f(x) = \log|x|$.