Example 1. Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent or divergent.

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Since $\frac{1}{x^2}$ is decreasing,

$$\frac{1}{2^2} \leq \int_1^2 \frac{1}{x^2} \, dx,$$

$$\frac{1}{3^2} \leq \int_2^3 \frac{1}{x^2} \, dx,$$

$$\frac{1}{4^2} \leq \int_3^4 \frac{1}{x^2} \, dx, \ldots$$
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\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\leq 1 + \int_1^2 \frac{1}{x^2} \, dx + \int_2^3 \frac{1}{x^2} \, dx + \int_3^4 \frac{1}{x^2} \, dx + \cdots
\]

\[
= 1 + \int_1^\infty \frac{1}{x^2} \, dx.
\]
Since \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) is convergent, the partial sum \( s_n \) is bounded. Also, \( a_n > 0 \) implies \( s_n \) is increasing.

Recall Fact 5: Every bounded monotonic sequence is convergent.

Thus \( s_n \)'s limit exists, i.e. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent.
This example shows us that the convergence of $\int_1^{\infty} \frac{1}{x^2} \, dx$ implies convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This can be generalized to a large class of series:
The integral test Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) \, dx$ is convergent.
Example 2. For what values of $p$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution: Use the integral test. We know from Chapter 7.8 Indefinite Integral that $\int_1^{\infty} \frac{1}{x^p} \, dx$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Conclusion: The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$. 
Example 3. Determine if the series \( \sum_{n=1}^{\infty} \frac{1}{n^3+4} \) converges or diverges.

Solution: Use the integral test.

Consider \( f(x) = \frac{1}{x^3+4} \). It is positive, continuous and decreasing. Why is it decreasing? Because \( x^3 + 4 \) is increasing.
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Now

\[ \int_{1}^{\infty} \frac{1}{x^3 + 4} \, dx \leq \int_{1}^{\infty} \frac{1}{x^3} \, dx. \]

By Chapter 7.8, the indefinite integral on the right hand side is convergent. Thus \( \int_{1}^{\infty} \frac{1}{x^3 + 4} \, dx \) is also convergent. So the series is convergent.