Suppose that $x, y$ are both given as functions of a third variable $t$ (called a parameter)

$$x = f(t), \quad y = g(t),$$

where $t \in (a, b)$.

- Parametric equations.
- As $t$ varies, the collection of points $(x(t), y(t))$ form a curve. We call it parametric curve.
Some parametric curves can be written in Cartesian equation (i.e. uses only $x$ and $y$ without introducing the parameter $t$).

Example 1. $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$ is a parametric curve.

Indicate with an arrow the direction in which the curve is traced as the parameter increases.

Eliminate the parameter to find a Cartesian equation of the curve.

Indicate with an arrow the direction in which the curve is traced as the parameter increases.
Solution: Our goal is to eliminate the variable $t$. Each point $(x(t), y(t))$ satisfies

$$x^2 + y^2 = 1.$$ 

Thus it is on the circle. Also, every point on the circle corresponds to a point $(x(t), y(t))$ for some $t \in [0, 2\pi]$. 

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Parametric curves

In some cases, we can also transform the Cartesian equation to the parametric equations.

Example 2. Write the parabola $y = x^2$ in parametric equations in $t$.

Remark: There are many parametric equations that satisfies $y = x^2$. We only need to find one of them.

Solution: $x = t$, $y = t^2$, $t \in (-\infty, \infty)$. 
Example 3. $x = \frac{1}{2} \cos \theta, y = 2 \sin \theta, 0 \leq \theta \leq \pi$. Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.
Solution:

Notice

\[ 4x^2 + \frac{y^2}{4} = 1. \]

This is equation of an ellipse. See the picture.

Since \(0 \leq \theta \leq \pi\), \(y \geq 0\).

This is not the whole ellipse, but only the upper half of the ellipse.
Example 4. Find the parametric equation for the circle centered at $(1, 2)$ with radius 2.

Solution: $x = 1 + 2 \cos t$, $y = 2 + 2 \sin t$, $t \in [0, 2\pi]$. 