Geometric sequence

\[ \lim_{n \to \infty} r^n = 0 \text{ if } |r| < 1. \]
\[ \lim_{n \to \infty} r^n = \infty \text{ if } |r| > 1. \]
\[ \lim_{n \to \infty} (-1)^n \text{ does not exist.} \]
\[ \lim_{n \to \infty} 1^n = 1. \]

**Conclusion:** The sequence \( \{r^n\} \) converges if \(-1 < r \leq 1\), and it diverges for all other values of \( r \).
Sequences

Definition

If $a_1 > a_2 > a_3 > \cdots$, we say the sequence is decreasing.
If $a_1 < a_2 < a_3 < \cdots$, we say the sequence is increasing.
We say a sequence is monotonic if it is either increasing or decreasing.
Example 8. Show that the sequence $a_n = \frac{n}{n^2 + 1}$ is decreasing.

Solution: Note that for $f(x) = \frac{x}{x^2 + 1}$, $f'(x) = \frac{1-x^2}{(x^2+1)^2}$. It is negative when $x^2 > 1$. Thus $f$ is decreasing. Hence for $n \geq 1$, the sequence is decreasing.
Definition

If there is a constant $M$, so that $a_n \leq M$ for all $n$, then we say the sequence is bounded from above.

If there is a constant $m$, so that $m \leq a_n$ for all $n$, then we say the sequence is bounded from below.

$M$ is an upper bound for $\{a_n\}$. $m$ is a lower bound for $\{a_n\}$. 
Fact 5: Every bounded, monotonic sequence is convergent.

Example 9 (Example 14 on Page 703): Investigate the sequence \( \{a_n\} \) defined by the recurrence relation
\[
a_1 = 2, \quad a_{n+1} = \frac{1}{2} (a_n + 6)
\]
for \( n = 1, 2, 3, \ldots \).
Solution: We compute the first few terms and try to see the pattern.

\[ a_1 = 2, \ a_2 = 4, \ a_3 = 5, \ a_4 = 5.5, \ a_5 = 5.75, \ a_6 = 5.875, \ldots \]

It suggests it is an increasing sequence, and the terms are approaching 6.
Sequences

Prove the sequence is increasing by induction.

1) \( a_2 = 4 > a_1 \).

2) Suppose \( a_{n+1} > a_n \) is valid for \( n = k \), it is also valid for \( n = k + 1 \).

Prove the sequence is bounded by 6 by induction.

1) \( a_1 = 2 < 6 \).

2) Suppose \( a_n < 6 \) is valid for \( n = k \), it is also valid for \( n = k + 1 \).
Therefore \( \{a_n\} \) is an increasing and bounded sequence. By Fact 5, it is convergent, i.e. \( \lim_{n \to \infty} a_n \) exists.

How to determine \( \lim_{n \to \infty} a_n \)? (Fact 5 does not determine the value of the limit.)

Denote

\[
\lim_{n \to \infty} a_n = L.
\]

Using recurrence formula,

\[
\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2} (a_n + 6) = \frac{1}{2} (\lim_{n \to \infty} a_n + 6)
\]

Thus \( L = \frac{1}{2} (L + 6) \). This implies \( L = 6 \).
Series

Definition

\[ a_1 + a_2 + a_3 + \cdots \] is called an infinite series or just series.

Denoted by

\[ \sum_{n=1}^{\infty} a_n, \text{ or } \sum a_n. \]
Given a series $\sum_{n=1}^{\infty} a_n$, let $s_n$ denote its partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$ 

If $\lim_{n \to \infty} s_n$ exists as a finite number, then the series

$$\sum_{n=1}^{\infty} a_n := \lim_{n \to \infty} s_n,$$

and we say it is **convergent**.

If $\lim_{n \to \infty} s_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ is **divergent**.
Example 1. Suppose $s_n = \frac{3n}{2n+3}$. Then

\[
a_n = s_n - s_{n-1} = \frac{3n}{2n+3} - \frac{3(n-1)}{2(n-1)+3} = \frac{3n}{2n+3} - \frac{3n-3}{2n+1}.
\]

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3n}{2n+3} = \frac{3}{2}.
\]
Example 2. Geometric series $a_n = a \cdot r^{n-1}$. Compute $\sum_{n=1}^{\infty} a_n$ for $-1 < r < 1$.

Remark: in other words, $\{a_n\}$ is a geometric series if $\frac{a_{n+1}}{a_n} = r$ for all $n$.

Solution:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}, \quad (1)$$

for $r \neq 1$. 


How to prove identity (1)?

\[ s_n = a + ar + ar^2 + \cdots + ar^{n-1} \]

\[ r \cdot s_n = ar + ar^2 + ar^3 + \cdots + ar^n \]

Thus

\[ (1 - r)s_n = a - ar^n \]

which is equivalent to

\[ s_n = a \frac{1 - r^n}{1 - r} \]

for \( r \neq 1 \).
If $-1 < r < 1$, then $r^n \to 0$. Thus

\[ \lim_{n \to \infty} s_n = a \frac{1}{1 - r}. \]
If $r = -1$, then limit of $s_n = a \frac{1-r^n}{1-r}$ does not exist.

If $r = 1$, then the partial sum $s_n$ is not equal to $a \frac{1}{1-r}$. It should be $s_n = na$ whose limit is infinity.

If $r \leq -1$ or $r > 1$, then limit of $r^n$ does not exist. Hence limit of $s_n$ does not exist.
Conclusion: If $-1 < r < 1$, then

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = a \frac{1}{1 - r}.$$ 

The series converges.

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.
Example 3. Find the sum of geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Solution: Note $\frac{a_{n+1}}{a_n}$ is a constant for all $n$, and they all equal to $-\frac{2}{3}$. Thus it is a geometric series. $a = 5$, $r = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = -\frac{2}{3}$. Apparently, $|r| < 1$.

Thus the series equals to

$$a \frac{1}{1 - r} = 5 \cdot \frac{1}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{1 + \frac{2}{3}} = 3.$$
Example 4. Is the series $\sum_{n=1}^{\infty} 5^{2n} 2^{1-n}$ convergent or divergent?

Solution:

$$\sum_{n=1}^{\infty} 5^{2n} 2^{1-n} = \sum_{n=1}^{\infty} (5^2)^n \cdot 2 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{25}{2}\right)^n.$$ 

Note $r = \frac{25}{2} > 1$, thus the geometric series diverges.
Example 5. Find $\sum_{n=2}^{\infty} 2^{-n}$.

Solution:

$$\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{4} (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}.$$

Thus

$$\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}.$$