Series

Definition

\[ a_1 + a_2 + a_3 + \cdots \] is called an infinite series or just series.

Denoted by

\[ \sum_{n=1}^{\infty} a_n, \text{ or } \sum a_n. \]
Given a series \( \sum_{n=1}^{\infty} a_n \). The **partial sum** is the sum of the first \( n \) terms of the series, denoted by \( s_n \).

\[
    s_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.
\]

If \( \lim_{n \to \infty} s_n \) exists as a finite number, then the series

\[
    \sum_{n=1}^{\infty} a_n := \lim_{n \to \infty} s_n,
\]

and we say it is **convergent**.

If \( \lim_{n \to \infty} s_n \) does not exist, we say \( \sum_{n=1}^{\infty} a_n \) is **divergent**.
Example 1. Suppose \( s_n = \frac{3n}{2n+3} \). Then by definition,

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3n}{2n+3} = \frac{3}{2}.
\]
Example 2. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. And it is equal to 1.

Solution: The partial sum

$$s_n = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)}.$$ 

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

All the terms, except 1 and $-\frac{1}{n+1}$, cancel. So $s_n = 1 - \frac{1}{n+1}$.

Hence

$$\lim_{n \to \infty} s_n = 1.$$
By definition, it means $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and it is equal to 1.

Very few series have such a beautiful cancellation formula. For most of the series, we cannot compute the partial sum explicitly.
Example 3. (Geometric series) Compute $\sum_{n=0}^{\infty} a \cdot r^n$. For what values of $r$, this series converges.

Remark: Recall $\{a_n\}$ is a geometric sequence if $\frac{a_{n+1}}{a_n}$ equals a constant $r$, for all $n$.

Solution: If $r \neq 1$,

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a \frac{1 - r^n}{1 - r}. \quad (1)$$

When $r = 1$, $s_n = a + a + \cdots a = na$. 


Series

How to prove identity (1)?

\[ s_n = a + ar + ar^2 + \cdots + ar^{n-1} \]

\[ r \cdot s_n = ar + ar^2 + ar^3 + \cdots + ar^n \]

Subtract the first line by the second line gives

\[(1 - r)s_n = a - ar^n \]

which is equivalent to

\[ s_n = a \frac{1 - r^n}{1 - r} \]

for \( r \neq 1 \).
If $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$. Thus

$$\lim_{n \to \infty} s_n = \frac{a}{1 - r}.$$
If $r = 1$, then the partial sum $s_n$ is not equal to $a \frac{1-r^n}{1-r}$. It is $s_n = na$, whose limit is infinity.

If $r \leq -1$ or $r > 1$, then limit of $r^n$ does not exist. (Recall the four cases in Chapter 11.1, in Oct 29 notes.) Hence limit of $s_n$ does not exist.
Conclusion: If $-1 < r < 1$, then the series converges and
\[ a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = a \frac{1}{1 - r}. \]

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges.
Example 4. Find the sum of geometric series

\[
5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots.
\]

Solution: Note \( \frac{a_{n+1}}{a_n} \) is a constant for all \( n \), and they all equal to \(-\frac{2}{3}\). Thus it is a geometric series. \( a = 5, \quad r = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = -\frac{2}{3} \).

Apparently, \(|r| < 1\).

Thus the series equals to

\[
a \frac{1}{1-r} = 5 \cdot \frac{1}{1 - \left(-\frac{2}{3}\right)} = 5 \cdot \frac{1}{1 + \frac{2}{3}} = 3.
\]
Example 5. Is the series $\sum_{n=1}^{\infty} 5^{2n} 2^{1-n}$ convergent or divergent?

Solution:

$$\sum_{n=1}^{\infty} 5^{2n} 2^{1-n} = \sum_{n=1}^{\infty} (5^2)^n \cdot 2 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{25}{2}\right)^n.$$ 

Note $r = \frac{25}{2} > 1$, thus the geometric series diverges.
Example 6. Find $\sum_{n=100}^{\infty} 2^{-n}$.

Solution:

$$\sum_{n=100}^{\infty} 2^{-n} = \left(\frac{1}{2}\right)^{100} + \left(\frac{1}{2}\right)^{101} + \left(\frac{1}{2}\right)^{102} + \left(\frac{1}{2}\right)^{103} + \cdots$$

$$= \left(\frac{1}{2}\right)^{100} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right)$$

$$= \left(\frac{1}{2}\right)^{100} \sum_{n=0}^{\infty} \frac{1}{2^n}.$$ (2)

$$|r| = \frac{1}{2} < 1,$$ thus $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$
Hence

$$\sum_{n=100}^{\infty} 2^{-n} = \left(\frac{1}{2}\right)^{100} \cdot \frac{1}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^{99}.$$

This example shows that we can compute geometric series starting with any index.
Theorem

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Remark: The converse is not true in general. If $\lim_{n \to \infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may be convergent and divergent. We will give examples in future.
Contrapositive Statement of the Theorem: If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

This is also called Divergence test.

Example 7. $\sum_{n=1}^{\infty} \frac{n}{2n+4}$ diverges.

Solution: Since $\lim_{n \to \infty} \frac{n}{2n+4} = \frac{1}{2} \neq 0$, by the contrapositive statement of the Theorem, $\sum_{n=1}^{\infty} \frac{n}{2n+4}$ diverges.