A first order linear differential equation is

\[ \frac{dy}{dx} + P(x)y = Q(x). \]

We call it linear, because \( \frac{dy}{dx} + P(x)y \) is linear in \( y \).

Namely, if \( y_1(x), y_2(x) \) are both solutions to \( \frac{dy}{dx} + P(x)y = 0 \), then \( ay_1(x) + by_2(x) \) is also a solution, for any constants \( a, b \).
Linear equations

Example 1. $xy' + y = 2x$. \[\Rightarrow y' + \frac{y}{x} = 2.\]

Example 2. When $P, Q$ are constants.

\[
\frac{dy}{dx} + Py = Q
\]

can be written as

\[
\frac{dy}{Q - Py} = dx.
\]
Linear equations

But when $P, Q$ are not constants, we need a different strategy to solve linear equations.

Example 3.

\[
\frac{dy}{dx} + y = x
\]

We multiply both sides by $e^x$. Then

\[
e^x \frac{dy}{dx} + ye^x = xe^x.
\]

Then

\[
(e^x \cdot y(x))' = xe^x.
\]
Linear equations

\[ \Rightarrow e^x \cdot y(x) = \int xe^x \, dx. \]

\[ \Rightarrow y(x) = e^{-x} \int xe^x \, dx. \]

Use integration by parts, we get the right hand side equals

\[ y(x) = e^{-x}(xe^x - e^x + C) = x - 1 + Ce^{-x}. \]
Linear equations

This indicates a general method to deal with linear equations.

► Step 0. Before applying the following steps, we need to change the equation into this standard form \( \frac{dy}{dx} + P(x)y = Q(x) \).

► Step 1. Write down factor \( I(x) := e^{\int P(x)dx} \) (we call it integrating factor), and multiply it on both sides so that

\[
\text{LHS} = I(x)\left(\frac{dy}{dx} + P(x)y\right) = (I(x)y(x))'
\]

This is because \( I(x)P(x) = I'(x) \)

► Step 2. We solve the equation

\[
(I(x)y(x))' = I(x)Q(x)
\]

by taking integration on both sides.
Step 3. We get

\[ I(x)y(x) = \int I(x)Q(x)dx. \]

Thus

\[ y(x) = \frac{1}{I(x)} \int I(x)Q(x)dx \]

is the general solution.
Example 4. \[ y' + 2xy = 1. \]

Solution: \( P(x) = 2x, Q(x) = 1. \) We take \( I(x) = e^{ \int P(x)dx} = e^{x^2}. \) Thus
\[
(e^{x^2} y)' = e^{x^2}.
\]
Integrate both sides, we get
\[
e^{x^2} y(x) = \int e^{x^2} dx + C.
\]
\(
\Rightarrow y(x) = e^{-x^2} \int e^{x^2} dx + Ce^{-x^2}.
\)

Example 5. \(2xy' + y = 2x. \) (Integrating factor is \( \frac{1}{\sqrt{x}}. \))