Calc I - Soln's Hw #10

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p. 287 #13:

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 2 \quad x \in \mathbb{R}$$

Find MAX/min

$$y' = x^2 + x - 6$$

$$y' = (x-2)(x+3) := 0 \quad \therefore \text{ critical pts: } x = 2, -3$$

$$y'' = 2x + 1$$

$$y''(2) = 5 \quad \therefore \boxed{\text{local min @ } 2 = x}$$

$$y''(-3) = -5 \quad \therefore \boxed{\text{local MAX @ } -3 = x}$$

F increasing when $f'(x) > 0$

$(x-2)(x+3) > 0$ when both pos or both negative

if both positive:

$x - 2 > 0$ & $x + 3 > 0$

$x > 2$ & $x > -3$

$\therefore$ $x > 2$

if both negative:

$x - 2 < 0$ & $x + 3 < 0$

$x < 2$ & $x < -3$

$\therefore$ $x < -3$

$\therefore$ $\boxed{\text{increasing on } (-\infty, -3) \cup (2, \infty)}$

$\therefore$ $\boxed{\text{decreasing on } (-3, 2)}$

p. 287 #16

$$y = \sqrt{1+x^2} = (1+x^2)^{1/2}$$

Note: $f(x)$ is an even function. If restrict domain to $\mathbb{R}^+_{>0}$, see that f is always increasing $\therefore$ for $\mathbb{R}^-_{<0}$, $f(x)$ always decreasing.

$$y' = \frac{1}{2}(1+x^2)^{-1/2}(2x) = x(1+x^2)^{-1/2} \quad \text{critical pt at } x = 0$$

$$y'' = x\left(\frac{-1}{2}\right)(1+x^2)^{-3/2}(2x) + (1+x^2)^{-1/2}$$

$y''(0)$ = positive $\therefore$ <u>local min</u>. By evenness of function $\Rightarrow$ <u>global min</u>

p. 287 #24

$f(x) = \ln x + \frac{1}{x} \quad x > 0$

$f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x^2 - x}{x^3} = \frac{x-1}{x^2} \quad \therefore$ cp at $x = 1$

$f''(x) = \frac{x^2 - (x-1)(2x)}{x^4}$

$f''(1) = 1 \quad \therefore$ local min

inflection points when $f''(0) \Rightarrow x^2 - 2x^2 + 2x = -x^2 + 2x = 0$

$= x(-x+2) \quad \therefore x = 0$ or $x = 2$

But $x = 0$ not in domain, so only inflexion point is $\boxed{x = 2}$

p. 287 #31

$y = x + \cos x$

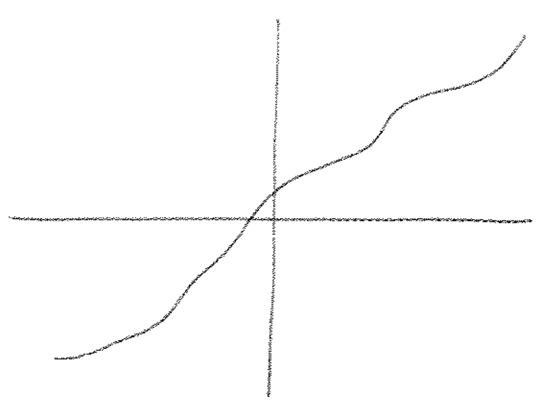
MAX/min: $y' = 1 \mp \sin x := 0 \quad x = \sin^{-1}(1) \quad \therefore$ critical pts at $x = \frac{(4k+1)\pi}{2}$ $k \in \mathbb{Z}$

$y'' = -\cos x$

$y'' = \begin{cases} > 0 & \frac{n\pi}{2} < x \quad \frac{3n\pi}{2} \\ < 0 & -\frac{n\pi}{2} < x < \frac{n\pi}{2} \end{cases} \quad n \in \mathbb{Z}_{>0}$

Inflection pt: $y'' = 0 \quad \cos x = 0 \quad \therefore x = \frac{(4k+1)\pi}{2} \quad k \in \mathbb{Z}$

Graph:



p. 288 #36

$$f(x) = \frac{-2}{x^2-1} = \frac{-2}{(x+1)(x-1)} \qquad x \neq 1, -1$$

a) $\lim_{x\to+\infty} f(x) = \frac{-2}{\infty} = 0$

$\lim_{x\to\infty} f(x) = \frac{2}{\infty} = 0$ $\quad \therefore y = 0$ is horizontal asymptote

b) $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{-2}{x^2-1} = \lim_{x\to1^+} \frac{-2}{(1+\varepsilon)^2-1} = -\infty$

$\lim_{x\to1^-} \frac{-2}{x^2-1} = \frac{-2}{(1-\varepsilon)^2-1} = \infty$

$\therefore x = \pm 1$ vertical asymptote

Similar for other case

c) $y' = 2(2x) = \frac{4x}{(x^2-1)^2} := 0$ at $x = 0$

$y'' = \frac{(x^2-1)^2(8) - 4x(2)(x^2-1)(2x)}{(x^2-1)^4} > 0$ $\therefore$ local min at $x = 0$

at 0

for $x < 1$, $f'(x) < 0$ $\therefore f$ decreasing

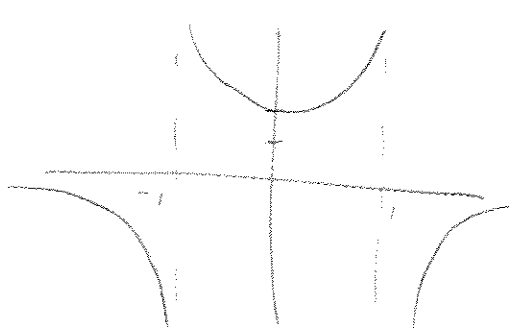
for $x > 1$, $f'(x) > 0$ $\therefore f$ increasing

for $x \in (-1, 0)$ $f'(x) < 0$ $\therefore f$ decreasing

for $x \in (0, 1)$, $f'(x) > 0$ $\therefore f$ increasing

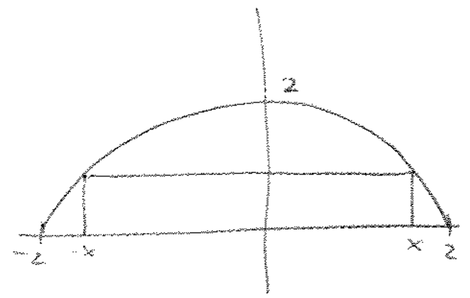
d) Concave up in $(-1, 1)$, down in $(-\infty, 1) \cup (1, \infty)$

e) Graph



p. 296 #10

$y = \sqrt{4-x^2}$ Find largest Area



Area of rectangle $= l \cdot w$

$= 2xy$

$= 2x(4-x^2)^{1/2}$

$A'(x) = 2x(\tfrac{1}{2})(4-x^2)^{-1/2}(-2x) + (4-x^2)^{1/2}\,2$

$= 2(4-x^2)^{-1/2}[-x^2 + 4 - x^2] := 0$

$=$

$4 - 2x^2 = 0$

$x = \sqrt{2}$

$\therefore y = \sqrt{4-2} = \sqrt{2}$

$\therefore$ Area is $A = 2xy = 2 \cdot 2 = \boxed{4}$

p. 296 #11

$y = 4 - 3x$

a) Distance to origin: $f(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + (4-3x)^2}$ by distance formula

b) Use square of distance: (Gives same as if not squared... just makes algebra easier)

$D^2 = x^2 + (4-3x)^2$

$D' = 2x + 2(4-3x)(-3)$

$= 2x - 24 + 18x = 20x \mp 24$

critical pt $x = \dfrac{24}{20} = \dfrac{6}{5}$ $y = \dfrac{2}{5}$

c) work it out to get same coordinates.

p. 297 #12

$y = 1/x$

$D^2 = x^2 + y^2$ ← distance to origin squared

$= x^2 + \frac{1}{x^2}$

$D'(x) = 2x - 2x^{-3}$

$\Rightarrow$ critical pts when $x^4 - 1 = 0$ $\therefore x = 1$ $\therefore y = 1$

$\therefore D = \sqrt{1+1} = \sqrt{2}$

p. 297 #14

let $f(x)$ differentiable w/ $f(x) < 0$ $\forall x \in \mathbb{R}$

let $f(c)$ be local max ($\therefore f'(c) = 0$, $f''(c) < 0$)

Show $g(x) = [f(x)]^2$ have local min

$g'(x) = 2f(x) \cdot f'(x)$

critical pt when $g'(x) = 0$, ie when $2f(x)f'(x) = 0$ ie when $f'(x) = 0$, so when $\underline{x = c}$

always neg, so divide out

classify critical pt at $x = c$:

$g''(x) = 2f(x)f''(x) + f'(x)(2f'(x)) = 2f(x)f''(x) + 2f'^2(x)$

$g''(c) = 2\underset{\text{neg}}{f(c)}\,\underset{\text{neg}}{f''(c)} + 2[\underset{\text{zero}}{f'(c)}]^2 = 2 \cdot \text{positive} = \text{positive}$ $\therefore$ $\boxed{\text{local min}}$

p. 307 #16

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} = \left(\frac{0}{0}\right) \overset{LHR}{=} \lim_{x\to 0} \frac{e^x - 1 + x}{3x^2} \quad \left(\frac{0}{0}\right)$$

$$\overset{LHR}{=} \lim_{x\to 0} \frac{e^x + 1}{6x} \quad \left(\frac{0}{0}\right) \overset{LHR}{=} \lim_{x\to 0} \frac{e^x}{6} = \boxed{\frac{1}{6}}$$