THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
MIDTERM EXAM - FALL SESSION 2007
110.106 - CALCULUS I.

Examiner: Professor C. Consani
Duration: 50 MINUTES (10am-10:50am), October 31, 2007.

No calculators, books, notes allowed. **Total Points = 100**

Student Name: ______________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature: ______________________

TA Name (circle one): J. Cutrone, T. Wright

1.	
2.	
3.	
4.	
5.	
Total	

1. [20 points] Compute $f'(x)$ for

$$f(x) = 3x^2 + 7, \quad f(x) = \sin(\frac{2}{x} - x), \quad f(x) = \frac{1}{2x^2+1}.$$

At which $x \in \mathbb{R}$ is $f'(x)$ defined, for each $f(x)$?

$$f(x) = 3x^2 + 7$$

$$f'(x) = 6x \qquad \forall x \in \mathbb{R}$$

$$f(x) = \sin\left(\frac{2}{x} - x\right)$$

$$f'(x) = \cos\left(\frac{2}{x} - x\right) \cdot \left(-\frac{2}{x^2} - 1\right) \qquad \forall x \in \mathbb{R} - \{0\}$$

$$f(x) = \frac{1}{2x^2+1}$$

$$f'(x) = \frac{-4x}{(2x^2+1)^2} \qquad \forall x \in \mathbb{R}$$

2. [15 points] Compute the following limits

$$\lim_{x\to 0} \frac{x^2-3x}{\sin(4x)}, \quad \lim_{x\to 0} \frac{\tan(3x)}{x^3+5x}, \quad \lim_{x\to\infty} \cos(\frac{1}{x}).$$

$$\lim_{x\to 0} \frac{x^2-3x}{\sin(4x)} = \lim_{x\to 0}\left(\frac{4x}{\sin(4x)} \cdot \frac{x^2-3x}{4x}\right) =$$

$$= \lim_{x\to 0} \frac{4x}{\sin(4x)} \cdot \left[\left(\lim_{x\to 0} \frac{x}{4}\right) - \frac{3}{4}\right] = 1\left[0 - 3/4\right] = -3/4$$

$$\lim_{x\to 0} \frac{\tan(3x)}{x^3+5x} = \lim_{x\to 0}\left(\frac{\sin(3x)}{\cos(3x)} \cdot \frac{1}{x^3+5x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{\sin(3x)}{3x} \cdot \frac{3x}{x^3+5x} \cdot \frac{1}{\cos(3x)}\right) =$$

$$= \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \lim_{x\to 0} \frac{3}{x^2+5} \cdot \lim_{x\to 0} \frac{1}{\cos(3x)} = 1 \cdot \frac{3}{5} \cdot 1 = 3/5$$

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$$

3. [20 points] Find the points (x, y) on the graph of the function

$$f(x) = x^3$$

at which the tangent line is horizontal and the points at which the tangent line is parallel to the line $y = x$: write the equation of these tangent lines.

The slope of the tangent line at a point $P = (x, f(x))$ is:

$$m(x) = f'(x) = 3x^2$$

The tangent line is horizontal when $m(x) = 0$ that is at $P = (0,0)$

The tangent line is parallel to the line $y = x$ if $m(x) = 1$: that is at the points $Q_1 = \left(\frac{1}{3}, \frac{1}{27}\right)$, $Q_2 = \left(-\frac{1}{3}, -\frac{1}{27}\right)$

The equation of the tg line at $P = (0,0)$ is

$$y = 0$$

The equation of the tg line at $Q_1 = \left(\frac{1}{3}, \frac{1}{27}\right)$ is

$$y - \frac{1}{27} = \left(x - \frac{1}{3}\right)$$

The equation of the tg line at $Q_2 = \left(-\frac{1}{3}, -\frac{1}{27}\right)$ is

$$y + \frac{1}{27} = \left(x + \frac{1}{3}\right)$$

4. [25 points] Determine for which n positive integer(s), the derivative $f'(x)$ exists at all real numbers and for which n positive integer(s) $f'(x)$ is continuous at all real numbers

$$f(x) = \begin{cases} x^n \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

$f(x)$ is differentiable if $x \neq 0$.

Therefore, the study of the differentiability of $f(x)$ is reduced to the study at $(0,0)$:

A necessary condition for the differentiability is that $f(x)$ must be continuous (at $(0,0)$): let's verify for which n this happens:

$$\lim_{x \to 0} x^n \sin\left(\frac{1}{x}\right) = f(0) = 0 \quad \text{if and only if } n \geq 1$$

in fact: if $n \geq 1$:

$$\underbrace{-x^n}_{x\to 0\ \to\ 0} \leq x^n \sin\left(\frac{1}{x}\right) \leq \underbrace{x^n}_{x \to 0\ \to\ 0}$$

Let's now study the differentiability of $f(x)$ at $O=(0,0)$.

$$\lim_{h \to 0} \frac{h^n \sin(\frac{1}{h})}{h} = \lim_{h \to 0} h^{n-1} \sin\left(\frac{1}{h}\right) = 0 \quad \text{if and only if } n \geq 2 \quad (\text{if } n=1\ f'(0)\ \nexists)$$

Let's now study the continuity of $f'(x)$ at $O=(0,0)$

($\forall x \neq 0$ $f'(x)$ is continuous)

$$f'(x) = \begin{cases} n x^{n-1} \sin(\frac{1}{x}) + x^n \cos(\frac{1}{x})(-\frac{1}{x^2}) & x \neq 0,\ n \geq 2 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0}\left(n x^{n-1} \sin\left(\frac{1}{x}\right) - \frac{1}{x^2} \cdot x^n \cos\left(\frac{1}{x}\right)\right) = 0 \iff n \geq 3$$

so $f'(x)$ is continuous $\forall x \in \mathbb{R} \iff n \geq 3$

5. [20 points] How many real roots does

$$x^3 - 4x + 2 = 0$$

have? Locate them between consecutive integers.

Let $f(x) = x^3 - 4x + 2$

$f(-3) = -27 + 12 + 2 < 0$
$f(-2) = -8 + 8 + 2 > 0$

The intermediate Value Thm assures that $\exists\ x_1 \in (-3,-2)$ st $f(x_1) = 0$

$f(-1) = -1 + 4 + 2 > 0$

$f(0) = 2 > 0$
$f(1) = 1 - 4 + 2 < 0$

The intermediate Value Thm. $\rightarrow \exists\ x_2 \in (0,1)$ st $f(x_2) = 0$

$f(2) = 8 - 8 + 2 > 0$

Again by applying the I.V.T. we conclude that $\exists\ x_3 \in (1,2)$ st $f(x_3) = 0$

Hence the polynomial $x^3 - 4x + 2$ has 3 real roots.