THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
MIDTERM EXAM - FALL SESSION 2007
110.106 - CALCULUS I.

Examiner: Professor C. Consani
Duration: 50 MINUTES (10am-10:50am), November 28, 2007.

No calculators, books, notes allowed. **Total Points = 100**

Student Name: ______________________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature: ___________________________

TA Name (circle one): J. Cutrone, T. Wright

1.	
2.	
3.	
4.	
Total	

SOLUTION

1. [25 points] Compute the following limits

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right), \qquad \lim_{x\to\infty} \frac{x+\cos x}{x}.$$

Sol. The first limit is computed using Hospital's rules: $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x}) = 0$, as

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1}{\frac{\sin x}{x} + \cos x} = 0.$$

For the computation of the second limit, one cannot apply Hospital's rule as we do not have an indeterminate form, however

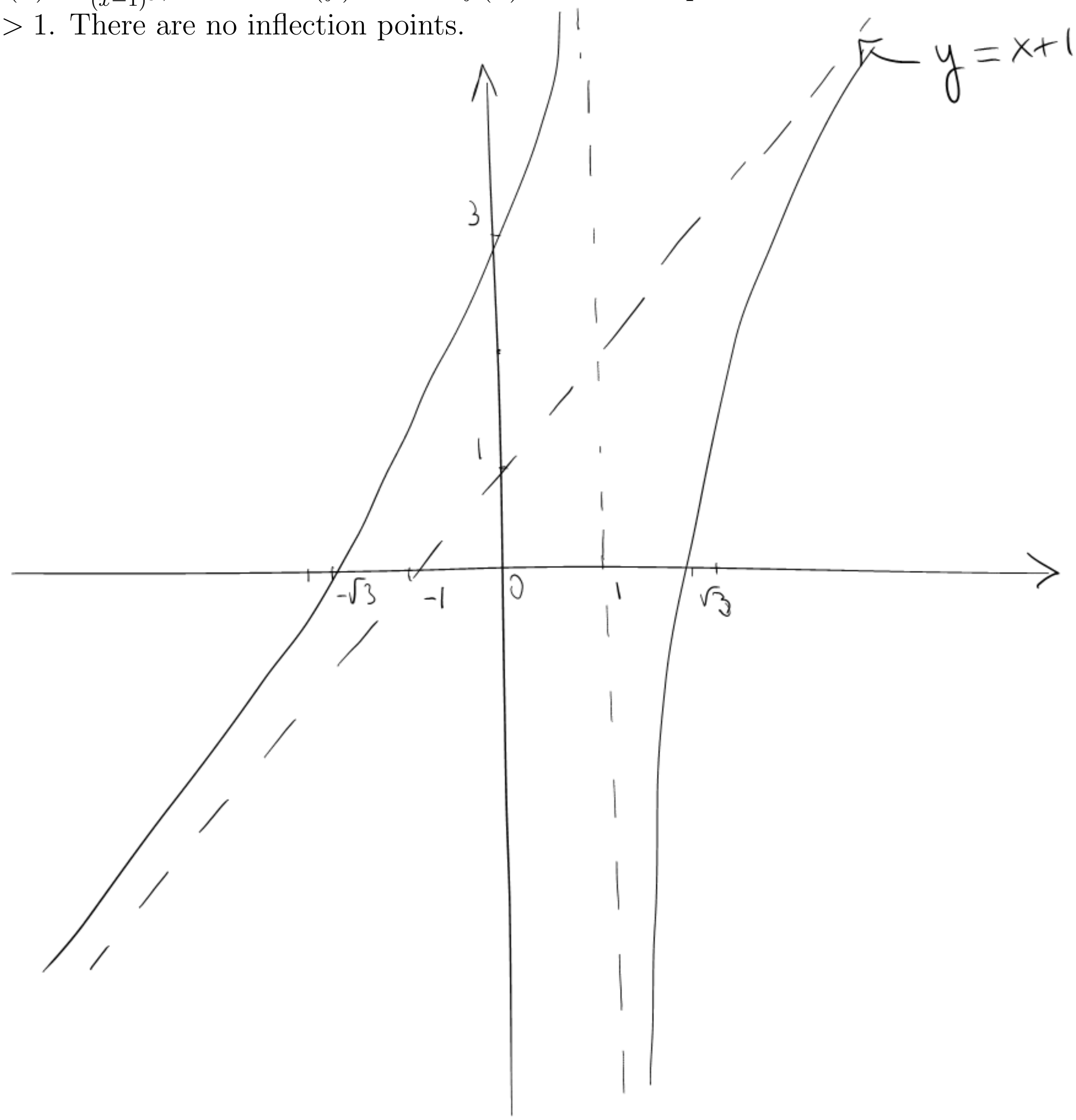
$$\lim_{x\to\infty} \frac{x+\cos x}{x} = \lim_{x\to\infty} \left(1 + \frac{\cos x}{x}\right) = 1$$

2. [30 points] Sketch the graph of the following function

$$f(x) = \frac{x^2 - 3}{x - 1}.$$

That means: determine the domain, the asymptotes, the local and the absolute extrema, the concavity and the inflection points. Finally, and based on these answers, you may sketch the graph.

Sol. $\text{Dom}(f) = \mathbb{R}\backslash\{1\}$. Hence $x = 1$ is the (unique) vertical asymptote. $\lim_{x\to\infty} \frac{x^2-3}{x(x-1)} = 1$, and $\lim_{x\to\infty}(f(x) - x) = \lim_{x\to\infty} \frac{x-3}{x-1} = 1$. Hence the line of equation $y = x + 1$ is the (unique) oblique asymptote.
$f'(x) = \frac{x^2-2x+3}{(x-1)^2} > 0$, $\forall x \in \text{Dom}(f)$: hence $f(x)$ is increasing and there aren't local or absolute extrema.
$f''(x) = \frac{-4}{(x-1)^3}$, $\forall x \in \text{Dom}(f)$. Hence $f(x)$ is concave-up when $x < 1$ and down when $x > 1$. There are no inflection points.



3. [25 points] A farmer has enough material to make a fence of length L. He wants to fence part of an open field in such a way as to contain the largest possible area within a rectangular region (of perimeter L) formed by the fence. Prove that a square shape gives the largest area.

Sol. $x, y =$ sides of the rectangle: $L/2 = x + y$. Thus $y = L/2 - x$.
Area of the rectangle: $A(x) = x(L/2 - x) = -x^2 + L/2x$.
Maximize the function $A(x) = -x^2 + L/2x$. Find $A'(x)$ and set $A'(x) = 0$ gives $x = L/4$. Max or min? $x = L/4$ is the first coordinate of the vertex of the (concave down) parabola $y = A(x)$. Therefore $x_{max} = L/4$ is an absolute maximum. Since $y_{max} = L/4 = x_{max}$ it follows that the square shape gives the largest area ($A(L/4) = L^2/16$).

4. [20 points] How many intercepts are there between the graphs of the following two functions:

$$f(x) = x^2, \qquad g(x) = x \sin x + \cos x.$$

Hint: consider the function $F(x) = f(x) - g(x)$: is it even/odd? What that does imply in the study of the number of intercepts? If one assumes the existence of $x_1,\ x_2 > 0$, such that $F(x_1) = 0 = F(x_2)$, what that does imply?

Sol. $F(-x) = F(x)$ so $F(x)$ is even, therefore the number of intersection points is even. $F(0) < 0$, $F(\pi) > 0$ hence by the Intermediate Value Theorem we have at least 2 intersection points. Finally, Rolle's Theorem applied in any interval $[x_1, x_2]$, with $x_1, x_2 > 0$ and $F(x_1) = 0 = F(x_2)$, implies that $F'(c) = 0$, for some $c \in (x_1, x_2)$. But this is impossible as $F'(x) = x(2 - \cos x) > 0$.