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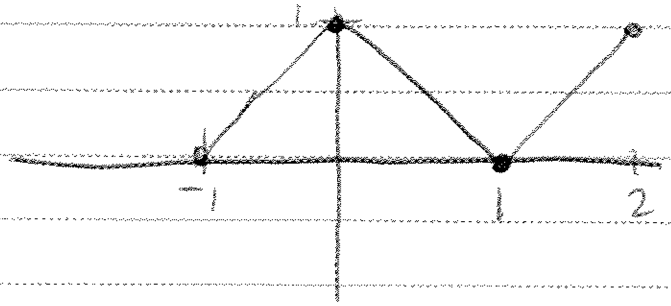

Local max: (0,1), (2,1)
Global max: (0,1), (2,1)
Local min: (-1,0), (1,0)
Global min: (-1,0), (1,0)

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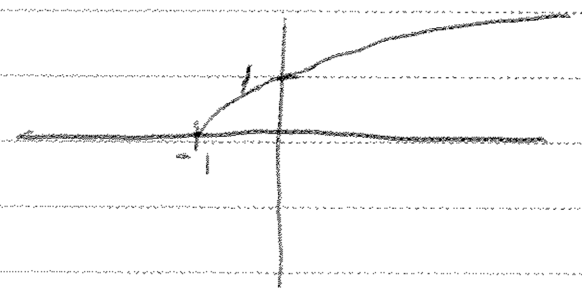
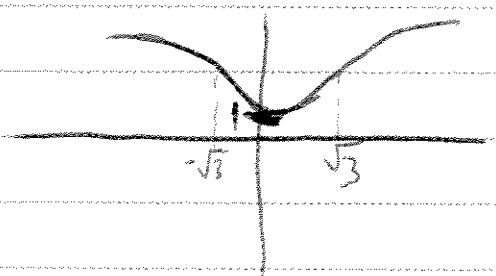


11



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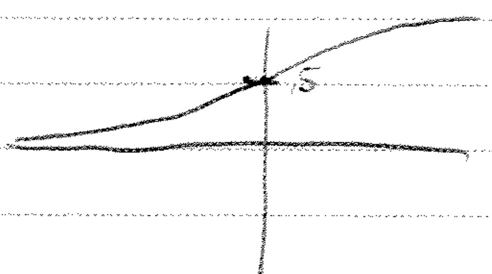

35.) a.) Slope of secant line= $\frac{f(2)-f(0)}{2-0} = \frac{4-0}{2-0} = 2$

b.) By MVT, we know that there must exist a c such that $f'(c)$ is equal to the slope of the secant line.

$f(x) = x^2$
$f'(x) = 2x$
$f'(c) = 2c = 2$
$c = 1$

37.) Since $f(-1) = f(1)$, by Rolle's Theorem there must exist a c between -1 and 1 such that $f'(c) = 0$ (i.e. the tangent line is horizontal).

$f'(x) = 2x$
$f'(c) = 2c = 0$
$c = 0$

40.) If we can find a and b such that $f(a) = f(b)$ then, MVT, there exists c such that

$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$

Any answer where $a = -b$ works, since

$f(-b) = \frac{1}{(-b)^2+1} = \frac{1}{b^2+1} = f(b)$

For example, if we take $a = -1$, $b = -1$ then

$f(-1) = \frac{1}{2}$
$f(1) = \frac{1}{2}$

So there exists c such that

$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{2}-\frac{1}{2}}{1-(-1)} = 0$

So $[-1, 1]$ is one possible solution.

47.) We know that $f(a) = f(b) = 0$. Since the function is not constant, there must be a c between a and b (i.e. $a < c < b$) such that $f(c) \neq 0$. Then, by MVT, along the interval $[a, c]$ there must be a c_1 such that

$f'(c_1) = \frac{f(c)-f(a)}{c-a} = \frac{f(c)}{c-a}$

Likewise, along the interval $[c, b]$, by MVT there must be a c_2 such that

$$f'(c_2) = \frac{f(b)-f(c)}{b-c} = \frac{-f(c)}{b-c}$$

Clearly, $\frac{f(c)}{c-a}$ and $\frac{-f(c)}{b-c}$ have opposite signs (since the denominators are both positive). So one of these is positive and one is negative, which is as required.

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7.) See graph page for a sketch

$$y = \sqrt{x+1}$$
$$y' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$$
$$y'' = -\frac{1}{4}(x+1)^{-\frac{3}{2}}$$

Note that y' is always positive for $x > -1$ and y'' is always negative for $x > -1$. So the graph is always increasing and concave down.

11.) See the graph page for a sketch

$$y = (x^2+1)^{\frac{1}{3}}$$
$$y' = \frac{2}{3}x(x^2+1)^{-\frac{2}{3}}$$

$$\begin{aligned} y'' &= \tfrac{2}{3}x(-\tfrac{2}{3})(x^2+1)^{-\frac{5}{3}}(2x) + \tfrac{2}{3}(x^2+1)^{-\frac{2}{3}} \\ &= \tfrac{2}{3}x(-\tfrac{2}{3})(x^2+1)^{-\frac{5}{3}}(2x) + \tfrac{2}{3}(x^2+1)\cdot(x^2+1)^{-\frac{5}{3}} \\ &= (\tfrac{2}{3}x(-\tfrac{2}{3})(2x) + \tfrac{2}{3}(x^2+1))(x^2+1)^{-\frac{5}{3}} \\ &= (-\tfrac{8}{9}x^2 + \tfrac{2}{3}x^2 + \tfrac{2}{3}))(x^2+1)^{-\frac{5}{3}} \\ &= (-\tfrac{8}{9}x^2 + \tfrac{2}{3}x^2 + \tfrac{2}{3}))(x^2+1)^{-\frac{5}{3}} \\ &= (-\tfrac{2}{9}x^3 + \tfrac{6}{9})(x^2+1)^{-\frac{5}{3}} \\ &= \tfrac{2}{9}(-x^3+3)(x^2+1)^{-\frac{5}{3}} \end{aligned}$$

For the y', note that $(x^2+1)^{-\frac{2}{3}}$ part is always positive. So y' is positive if and only if $\frac{2}{3}x$ is positive, which happens when $x > 0$. Thus, y is increasing when $x > 0$ and decreasing when $x < 0$.

For concavity, note that $(x^2+1)^{-\frac{5}{3}}$ is always positive. So y'' is positive if and only if $(-x^2+3)$ is positive. Note that

$$-x^2 + 3 = 0 \Rightarrow x = \pm\sqrt{3}$$

So we check the intervals between the zeroes. To the left of $x = -\sqrt{3}$, y'' is negative (to check, just plug a number less than $-\sqrt{3}$ in for x in the y'' equation; $x = -2$ is one possibility). Between $x = -\sqrt{3}$ and $x = \sqrt{3}$, y'' is positive (you can check by plugging in $x = 0$), and to the right of $x = \sqrt{3}$, y'' is negative.

Thus, y is concave up along $(-\sqrt{3}, \sqrt{3})$ and negative for $(-\infty, -\sqrt{3}) \bigcup (\sqrt{3}, \infty)$.

20.) See the graph page for a sketch

$$y = \frac{1}{1+e^{-x}}$$
$$y' = \frac{e^{-x}}{(1+e^{-x})^2}$$
$$y'' = \frac{(1+e^{-x})^2(-e^{-x}) - e^{-x}2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4}$$

Now, for y', if $y' = 0$,

$$\frac{e^{-x}}{(1+e^{-x})^2} = 0$$
$$e^{-x} = 0$$

where the last step occurs because we multiply both sides by $(1 + e^{-x})^2$. Note that e^{-x} can never equal zero, so the first derivative is always positive or always negative. To figure out which, we can plug in any value for x. Let's plug in $x = 0$; then $y' = \frac{1}{2} > 0$. So the derivative is always positive, and hence y is increasing everywhere.

For y'', find when $y'' = 0$:

$$\frac{(1+e^{-x})^2(-e^{-x}) - e^{-x}2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} = 0$$
$$(1 + e^{-x})^{-2}(-e^{-x}) - e^{-x}2(1 + e^{-x})^{-3}(-e^{-x}) = 0$$

Multiply both sides by $(1 + e^{-x})^3$. Then

$$(1 + e^{-x})(-e^{-x}) - e^{-x}2(-e^{-x}) = 0$$
$$-(1 + e^{-x})(e^{-x}) + 2e^{-x}(e^{-x}) = 0$$
$$-e^{-x} - e^{-2x} + 2e^{-2x} = 0$$
$$-e^{-x} + e^{-2x} = 0$$
$$e^{-x}(-1 + e^{-x}) = 0$$

Since e^{-x} is never zero, this equality means that

$$-1 + e^{-x} = 0$$
$$x = 0$$

So $y'' = 0$ when $x = 0$. To the left of $x = 0$, y'' is always positive (check a value for x less than 0 to show this), and to the right, y'' is always positive. Thus, y is concave up when $x < 0$ and concave down when $x > 0$.

23.) a.) Say the $f(x)$ has two roots at r_1 and r_2. Then, by Rolle's Theorem, $f'(x)$ must be zero somewhere between the two roots. But this is impossible because $f'(x)$ is strictly positive or negative.

b.) $f(-1) = 4$ and $f(1) = -2$. So by IVT, there exists a root between -1 and 1. Now,

$f'(x) = 3x^2 - 4$

If $f'(x)$ is zero,

$3x^2 - 4 = 0$
$x^2 = \frac{4}{3}$
$x = \pm\frac{4}{3}$

Since $\pm\frac{4}{3}$ is outside the interval $[-1, 1]$, $f'(x)$ is never zero along this interval. So it is strictly positive or strictly negative.

Thus, by part a, there is exactly one solution.

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10.) $f(x) = \ln(x^2 + 1) - x$

$f'(x) = \frac{2x}{x^2+1} - 1$

To find extrema, let $f'(x) = 0$. Then

$\frac{2x}{x^2+1} - 1 = 0$
$\frac{2x}{x^2+1} = 1$
$2x = x^2 + 1$
$0 = x^2 - 2x + 1$
$0 = (x - 1)^2$

So $x = 1$ is the only possible local extremum. To check, we use the second derivative test, using $x = 1$:

$f''(x) = \frac{(x^2+1)2-2x(2x)}{(x^2+1)^2}$
$f''(1) = \frac{(1^2+1)2-2(2)}{(1^2+1)^2} = \frac{4-4}{4} = 0$

So $x = 1$ is neither a min nor a max. So there are no local extrema. Since there are no endpoints and no local extrema, there are no global extrema either.

Since there are no minima or maxima, the function must always be increasing or decreasing. We can plug in any value for x to figure out whether the derivative is always positive or always negative. If we let $x = 0$, we get $f'(0) = -1$. So $f'(x)$ is always negative. Thus, the function is always decreasing.