THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
FINAL EXAM - FALL SESSION 2007
110.106 - CALCULUS I.

Examiner: Professor C. Consani
Duration: 3 HOURS (9am-12pm), December 13, 2007.

No calculators, books, notes allowed. **Total Points = 100**

Student Name: SOLUTIONS

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature: ______________________________

TA Name (circle one): J. Cutrone, T. Wright

1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total	

1. [10 points] Compute the following limits of sequences

$$\lim_{n\to\infty} \frac{n^2 - \sin n}{n^2 + n}; \qquad \lim_{n\to\infty} (\sqrt{2n+1} - \sqrt{2n-1}).$$

$$\lim_{n\to\infty} \frac{n^2 - \sin n}{n^2 + n} = \lim_{n\to\infty} \frac{n^2\left(1 - \frac{\sin n}{n^2}\right)}{n^2\left(1 + \frac{1}{n}\right)} = 1$$

$$\lim_{n\to\infty} \left(\sqrt{2n+1} - \sqrt{2n-1}\right) = \lim_{n\to\infty} \frac{2}{\sqrt{2n+1} + \sqrt{2n-1}} = 0$$

2. [15 points] Compute the following limits of functions

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1}; \qquad \lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}; \qquad \lim_{x\to 0} \frac{1}{x}(1-\frac{x}{\cos x}).$$

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = 2/3$$

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0} \left(\frac{\sin(2x)}{2x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{2x}{3x} \right) = 2/3$$

$\lim_{x\to 0} \frac{1}{x}(1-\frac{x}{\cos x}) \nexists$ in fact

$\lim_{x\to 0+} \frac{1}{x}(1-\frac{x}{\cos x}) = \infty$ and $\lim_{x\to 0-} \frac{1}{x}(1-\frac{x}{\cos x}) = -\infty$

3. [15 points] For each of the following functions $f(x)$ define the set where $f(x)$ is differentiable (*i.e.* where $f'(x)$ exists) and compute $f'(x)$; finally determine the set where $f'(x)$ is differentiable and compute $f''(x)$

$$f(x) = x|x| + x^2; \qquad f(x) = \begin{cases} x^4 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$f(x) = x|x| + x^2 = \begin{cases} 2x^2 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \mathrm{Dom}(f) = \mathbb{R}$$

$$\lim_{h \to 0} \frac{2h^2}{h} = 0 \Rightarrow f'(0) = 0 \quad \text{and} \quad f'(x) = \begin{cases} 4x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathrm{Dom}(f') = \mathbb{R}$$

$$\lim_{h \to 0+} \frac{4h}{h} = 4 \neq \lim_{h \to 0-} \frac{0}{h} = 0 \Rightarrow f''(0) \nexists \quad \text{and}$$

$$f''(x) = \begin{cases} 4 & x > 0 \\ 0 & x < 0 \end{cases} \qquad \mathrm{Dom}(f'') = \mathbb{R} - \{0\}$$

$$f(x) = \begin{cases} x^4 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \mathrm{Dom}(f) = \mathbb{R}$$

$$\lim_{h \to 0} \frac{h^4 \sin\left(\frac{1}{h}\right)}{h} = 0 \Rightarrow f'(0) = 0$$

$$f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \mathrm{Dom}(f') = \mathbb{R}$$

$$\lim_{h \to 0} \frac{4h^3 \sin\left(\frac{1}{h}\right) - h^2 \cos\left(\frac{1}{h}\right)}{h} = 0 \Rightarrow f''(0) = 0 \quad \text{and}$$

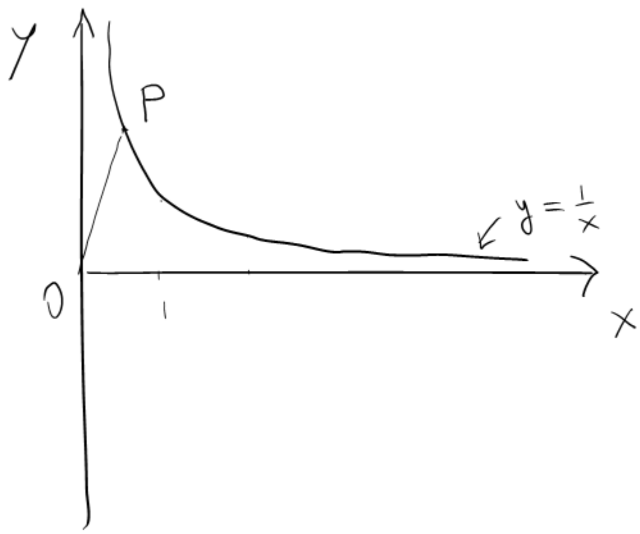
$$f''(x) = \begin{cases} 12x^2 \sin\left(\frac{1}{x}\right) - 6x \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \mathrm{Dom}(f'') = \mathbb{R}$$

4. [10 points] Find the point on the curve

$$y = \frac{1}{x}, \quad \text{for} \quad x > 0,$$

that is closest to the origin.

Hint Compute the absolute minimum of the function $f(x) = d^2(x)$, where $d(x)$ is the function that defines the distance between a point P on the graph of the curve and the origin $0 = (0, 0)$.



$d(x) = \text{LENGTH}(PO)$

$P = (x, \frac{1}{x})$

$f(x) = x^2 + \frac{1}{x^2}$ Dom $(f) = \mathbb{R} - \{0\}$ but we restrict the study on $(0, \infty)$

$f'(x) = 2x - \frac{2}{x^3}$ $\forall x \in (0, \infty)$

$f'(x) = 0 \iff \frac{2x^4 - 2}{x^3} = 0 \iff x^4 = 1 \iff x = -1$ OR $x = 1$

$f''(x) = 2 + \frac{6}{x^4}$ $\forall x \in (0, \infty)$; $f''(1) > 0 \Rightarrow x = 1$ is a local minimum

Finally, the sign of $f'(x)$ shows that:

sign(f') $\quad - \quad + \quad$ (0 ... 1 ...)

$f(x)$ decreases on $(0, 1)$

$f(x)$ increases on $(1, \infty)$

$\lim_{x \to 0+} \frac{1}{x} = \infty$

Hence we conclude that $f(1) = 2$ is the absolute minimum

5. [10 points] Compute the following integrals

$$\int \frac{dx}{\sqrt{2-3x}}; \qquad \int_{-1}^{1} \sin^{-1}(x)\ dx.$$

$$\int \frac{dx}{\sqrt{2-3x}} \underset{\substack{u=2-3x \\ du=-3dx}}{=} -\frac{1}{3}\int \frac{du}{\sqrt{u}} = -\frac{1}{3}\cdot 2\sqrt{u} + C$$

$$= -\frac{2}{3}\sqrt{2-3x} + C$$

$C \in \mathbb{R}$

$$\int_{-1}^{1} \sin^{-1}(x)\, dx = 0 \qquad \text{as} \qquad \sin^{-1}(x) \text{ is an odd function}$$

Alternatively:

$$\int_{-1}^{1} \sin^{-1}(x)\, dx = x \sin^{-1}(x)\Big]_{-1}^{1} - \int_{-1}^{1} x \frac{dx}{\sqrt{1-x^2}} =$$

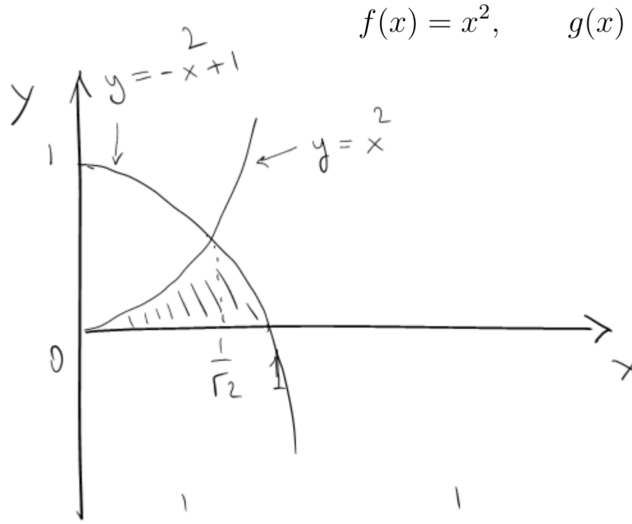
$$\underset{\substack{u=1-x^2 \\ du=-2x\,dx}}{=} \frac{1}{2}\int_{0}^{0} \frac{du}{\sqrt{u}} = 0$$

$x=-1 \to u=0$

$x=1 \to u=0$

6. [15 points] Find the area of the region bounded by the graphs of the following two functions and the X-axis, for $x \geq 0$

$$f(x) = x^2, \qquad g(x) = -x^2 + 1.$$



$$\text{Area} = \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} (-x^2+1)\,dx = \left.\frac{x^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} + \left(-\frac{x^3}{3} + x\right)\Big|_{\frac{1}{\sqrt{2}}}^{1} =$$

$$= \frac{1}{6\sqrt{2}} + \left[-\frac{1}{3} + 1 - \left(-\frac{1}{6\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{3\sqrt{2}} + \frac{2}{3} - \frac{1}{\sqrt{2}} =$$

$$= -\frac{2}{3\sqrt{2}} + \frac{2}{3} = \frac{2}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{2}{3}\,\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{\sqrt{2}}{3}(\sqrt{2}-1)$$

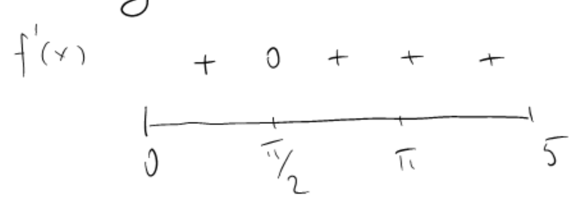
7. [15 points] Compute relative and absolute extrema (*i.e.* relative and absolute min/max) and inflection points of

$$f(x) = x + \cos x, \qquad \text{for} \quad x \in [0, 5].$$

$f(x) = x + \cos x \qquad x \in [0,5]$

$f'(x) = 1 - \sin x \qquad f'(x) = 0 \iff x = \pi/2$ on $[0,5]$

From the study of the sign of $f'(x)$



we deduce that $x = \pi/2$ is not a relative extremum

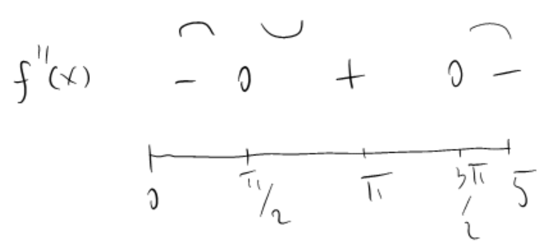
and there are no further relative extrema on $[0,5]$

Moreover $f'(x) \geq 0$ on $[0,5]$ $\Rightarrow$ $f(x)$ is increasing on $[0,5]$

So we conclude that $f(0) = 1$ is the absolute minimum and $f(5) = 5 + \cos 5$ is the abs. maximum

$f''(x) = -\cos x \qquad f''(x) = 0 \iff x = \pi/2, \ \frac{3}{2}\pi$

From the study of the sign of $f''(x)$



we deduce that $x = \pi/2$ and $x = 3/2\,\pi$ are inflection points

8. [10 points] Determine the number of real positive solutions (if any) of the following equation

$$e^x = x^2, \qquad x \in (0, \infty).$$

Justify mathematically your answer.

Let $g(x) = e^x$ and $h(x) = x^2$

$g'(x) = e^x$ and $h'(x) = 2x$

Is $e^x > 2x \quad \forall x > 0$?

Consider $f(x) = e^x - 2x$; we want to find the minimum

$f'(x) = e^x - 2 \qquad f'(x) = 0 \iff e^x = 2$ ie $x = \ln(2)$

$f'(x) \quad - \quad 0 \quad +$

$0 \qquad \ln(2)$

Hence $f(\ln(2)) = 2 - 2\ln(2)$ is the (relative and absolute) minimum

Notice that $2 - 2\ln(2) > 0$ (in fact $\ln(2) < 1$ as $e > 2$)

So $e^x > 2x \quad \forall x > 0 \implies$

$g'(x) > h'(x) \quad \forall x > 0$ (and $g(0) = 1 > h(0) = 0$)

$\implies (g(x) - h(x))' > 0 \quad \forall x > 0 \implies g(x) > h(x) \quad \forall x > 0$

Thus the equation $e^x = x^2$ has <u>No</u> solutions on $(0, \infty)$.