# MATH 106 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
Johns Hopkins University

May 6, 2004

NAME: ____________________________ SIGNATURE: ___________________________

SECTION NUMBER:_____________
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1. This exam has twelve pages including this cover. There are twelve questions.
2. Use of books, notes, or scratch paper is not allowed. You may certainly use a calculator (but not its manual).
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. Include units in your answers whenever appropriate. Showing your work will show graders that you are not using your calculator inappropriately. In general, you will not get credit for work done on your calculator if we expect you to do it by hand.
4. Read directions carefully. For some problems, a brief answer is sufficient, but others require you to show all work or give explanations.

| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 6 | |
| 2 | 5 | |
| 3 | 9 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 10 | |
| 7 | 6 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 8 | |
| 11 | 12 | |
| 12 | 8 | |
| TOTAL | 100 | |

**1.** (6 points) State each of the fundamental theorems of Calculus. Please use brief, symbolic, mathematical notation (as opposed to words and phrases) where appropriate.

**2.** (5 points) Calculate the derivative with respect to $x$ of

$$\frac{(x+5)(e^{2x}+x^2)}{\sin(x)}$$

**3.** (3 points each) Calculate these definite integrals by hand. Show your work.

**a.** $\int_0^\pi \cos(3x)dx$

**b.** $\int_3^4 \frac{1}{x}dx.$

**c.** $\int_9^{12} 2^x dx$

**4.** (8 points) A circular section is the shape bounded by two radii of a circle and the circular arc from one to another. (A piece of pie is usually a circular section.) A circular section with angle $\theta$ cut from a circle of radius $r$ has area $\frac{\theta}{2}r^2$. The length of the curved segment (just the “crust of the pie”) is $\theta r$. Find a circular segment whose perimeter (including all three edges) is 10 and whose area is as large as possible.

**5.** (8 points) The equation $xy^2 - y^4 = 4$ defines a curve in the plane, and the point $(5, 1)$ is on this curve.

**a.** (4 points) Find $\frac{dy}{dx}$ at this point.

**b.** (2 points) Find the equation of the tangent line to the curve at this point.

**c.** (2 points) What is the $y$ coordinate of a point on the curve whose $x$ coordinate is $5.08$? Estimate using part b.

**6.** (10 points) As you may or may not have noticed, Baltimore has weak water lines, and occasionally pipes break under the streets and sidewalks. When this happens, the water seems to bubble out of the street at an ever increasing rate. Assume...

1. The initial flow rate of the water is ten gallons/minute
2. The flow rate increases linearly (at a constant rate of increase) until it reaches $82$ gallons/minute, when the break is repaired.
3. The repair happens $48$ hours after the break.

How much water is wasted before the problem is solved? As usual, show all work.

**7.** (6 points) (This problem and the next are related, but mostly independent.) Use the method of substitution to calculate the integral below. Of course, show all work.

$$\int \frac{2x+3}{x^2+3x+2} dx$$

**8.** (10 points) (This problem and the previous are related, but mostly independent.)
**a.** (8 points) Use the method of partial fractions to calculate the integral below. Show all work.

$$\int \frac{2x+3}{x^2+3x+2}dx$$

**b.** (2 points) Is your result the same as that on the previous page? Justify.

**9.** (10 points)

**a.** (8 points) Compute the first four terms (constant term through cubic term) of the Taylor series (at $x = 0$, as usual) for the function below. Simplify your answer as much as possible.

$$f(x) = \frac{1}{(1-x)^2}$$

**b.** (2 points) Your answer to part a. can be used to estimate $\frac{1}{(1-0.1)^2}$ (without dividing, which can be arduous). What is this estimate? Is it close to the actual value?

**10.** (8 points) When my youngest cat, Lucy, feels excited, she starts running randomly, faster and faster, around the house, until she tires, and then she slows down. Of course, I have attempted to model her behavior mathematically, and I have come up with this formula for her velocity:

$$v(t) = 1/(1-t)^4$$

Here $t$ is time in seconds since she gets excited.

**a.** (6 points) According to my analysis, how far does Lucy run in the first two seconds after she gets excited?

**b.** (2 points) Does that sound reasonable? Explain.

**11.** (12 points) Here you will have to set up, *but not solve*, certain problems, by simply writing an integral which answers the question. Do not attempt to solve the integrals! For these problems, once you have written down the correct integral, you're done. In all these problems, "the curve" refers to the graph of $y = e^{-x^2}$ for *positive* values of $x$. Write an integral which calculates:

**b.** (3 points) The length of the curve.

**a.** (3 points) The area under the curve and over the $x$ axis.

**c.** (3 points) The volume obtained by rotating the area under the curve around the $x$ axis.

**d.** (3 points) The volume obtained by rotating the area under the curve around the $y$ axis.

**12.** (8 points) Using the table below, calculate as accurately as possible the integral $\int_2^6 xp''(x)dx$.

| $x$ | $p(x)$ | $p'(x)$ | $p''(x)$ |
|---|---|---|---|
| 1 | 12 | 1.5 | $-0.1$ |
| 2 | 14 | 1.3 | $-0.15$ |
| 3 | 15 | 1.2 | $-0.2$ |
| 4 | 17 | 1.1 | $-0.1$ |
| 5 | 18 | 1.0 | 0.9 |
| 6 | 19 | 2.3 | 1.1 |
| 7 | 24 | 3.4 | 1.5 |