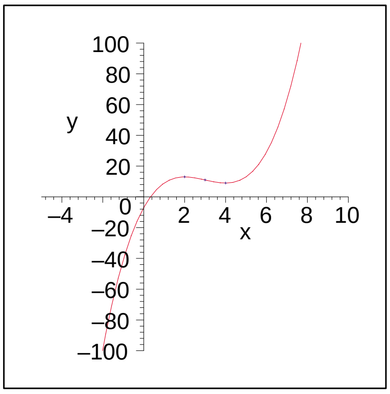
Solutions, Midterm 2, Math 106
Fall 2001, Professor Sogge

1) (20 pts.) Sketch the graph of $y = x^3 - 9x^2 + 24x - 7$. Plot any stationary points or points of inflection.

Solution: We first compute the first derivative, $y' = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-4)(x-2)$. We conclude that the function has $x = 4$ and $x = 2$ as its stationery points. Also, it is increasing on $(-\infty, 2)$ and $(4, +\infty)$, while it is decreasing on $(2, 4)$.

We next compute the second derivative $y'' = 6x - 18$, to conclude that $y'' < 0$ on $(-\infty, 3)$ and $y'' > 0$ on $(3, +\infty)$. Thus, $x = 3$ is an inflection point, and the function is concave down on $(-\infty, 3)$ and concave up on $(3, +\infty)$.

The graph of this function is below:



2) (15 pts.) Compute the following limits.

a) $\lim_{x \to \pi/2} \frac{(\frac{\pi}{2} - x)^2}{1 - \sin x}$

Solution: We need to use L'Hospital's rule since we are in the 0/0 situation. If we use L'Hospital twice we find that the limit equals

$$\lim_{x \to \pi/2} \frac{2(\frac{\pi}{2} - x)}{-\cos x} = \lim_{x \to \pi/2} \frac{-2}{-\sin x} = 2.$$

b) $\lim_{x \to 0+} \ln x \sin x$

Solution: Here we need to rewrite the expression and use L'Hospital's rule. By doing so, we find that the limit is

$$\lim_{x\to 0+} \frac{\ln x}{\csc x} = \lim_{x\to 0+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0+} \frac{-\sin x \tan x}{x}$$
$$= \lim_{x\to 0+} \frac{-\sin x}{x} \lim_{x\to 0+} \tan x = \lim_{x\to 0+} \frac{-\cos x}{1} \lim_{x\to 0+} \tan x = -1 \times 0 = 0.$$

c) $\lim_{x\to 0} \frac{1}{x^2} \int_0^x (e^{t^2} - 1)dt$

Solution: Here we need to use L'Hospital and the Fundamental Theorem of Calculus:

$$\lim_{x\to 0} \frac{\int_0^x (e^{t^2} - 1)dt}{x^2} = \lim_{x\to 0} \frac{(e^{x^2} - 1)}{2x} = \lim_{x\to 0} \frac{2xe^{x^2}}{2} = 0.$$

3) (15 pts.) Are the following true or false?

a) $f''(x) > 0$ on an open interval (a, b) then $f(x)$ is increasing on (a, b).

FALSE

b) If $f''(x) = 0$ then x is a point of inflection.

FALSE

c) If x_0 is a point of inflection then $f''(x_0) = 0$.

TRUE

4) (15 pts.) Express the number 27 as the sum of two nonnegative numbers whose product is as large as possible. *Show all of your work; partial credit will not be given if there is no work!*

Solution: Let x and y be two nonnegative numbers summing up to 27. Then $x + y = 27$. We wish to maximize their product, xy. Thus, since $y = 27 - x$, we wish to maximize the function $f(x) = x(27 - x)$ as x ranges over the interval $(0, 27)$. The derivative of this function is $f'(x) = -2x + 27$. It has $27/2$ as a stationary point, and the derivative is positive for $x < 27/2$ and negative for $x > 27/2$. Therefore, the function attains its maximum at the stationary point $x = 27/2$. Thus, since $y = 27 - 27/2 = 27/2$, the product is maximized when both x and y equal $27/2$.

5) (10 pts.) Evaluate the following integrals.

a) $\int (x^{13} + 13^x)dx$

Solution: If we write $13^x = e^{x \ln 13}$ we see that $\frac{d}{dx}\frac{13^x}{\ln 13} = 13^x$, and so $\frac{13^x}{\ln 13}$ is an antiderivative of 13^x. Therefore, the above integral equals

$$\frac{1}{14}x^{14} + \frac{13^x}{\ln 13} + C.$$

b) $\int_0^1 \frac{x^2}{1+x^2}\, dx$

Solution: If we recall that $\frac{d}{dx}\tan^{-1} x = 1/(1+x^2)$, and do some algebra, we find that the integral equals

$$\int_0^1 1\, dx - \int_0^1 \frac{1}{1+x^2}\, dx$$
$$= x - \tan^{-1} x]_0^1 = 1 - \tan^{-1} 0 + \tan^{-1} 1 = 1 - 0 + \frac{\pi}{4} = \frac{\pi}{4} + 1.$$

6) a) (10 pts.) Find a function y satisfying $\frac{dy}{dx} = \tan x \sec x$, $y(\pi/4) = \sqrt{2}$.

Solution: If we just use the fundamental theorem of calculus and integrate we see that the following is the function with this property:

$$\sqrt{2} + \int_{\pi/4}^{x} \tan t \sec t\, dt = \sqrt{2} + [\sec t]_{\pi/4}^{x}$$
$$= \sqrt{2} + \sec x - \sec(\pi/4) = \sqrt{2} + \sec x - \sqrt{2} = \sec x.$$

b) (5 pts.) Is there a function $f(t)$ so that $\int_1^x f(t)\, dt = x^2$? If so what is f? If there is no such f, explain why?

Solution: There is no such function. For if one plugs $x = 1$ into both sides, the integral vanishes at $x = 1$, while the right hand side, x^2, equals one there.

7) (10 pts.) Find the area of the region enclosed by the curves $y = (x-1)^2 - 1$ and $y = -x + 2$.

Solution: We first need to find where the two curves intersect. To do this we set $(x-1)^2 - 1 = -x+2$, which simplifies to $0 = x^2 - x - 2 = (x+1)(x-2)$. Thus, the two curves intersect at $x = -1$ and $x = 2$. On the interval $[-1, 2]$ the line $y = -x + 2$ is above the parabola $y = (x-1)^2 - 1$. Thus, the area of the region is

$$\int_{-1}^{2} \big((-x+2) - [(x-1)^2 - 1]\big) dx = \int_{-1}^{2} \big(2 + x - x^2\big)\, dx = 2x + \frac{x^2}{2} - \frac{x^3}{3}\Big]_{-1}^{2}$$
$$= (4 + 2 - \frac{8}{3}) - (-2 + \frac{1}{2} + \frac{1}{3}) = \frac{9}{2}.$$