ANSWERS

March 3, 2003

IN-CLASS MIDTERM I

Calculus 106 (Biological and Social Sciences) – Professor Haskins

Attempt all parts of all questions.
Remember to show all your work and to write clearly, neatly.
There are 5 questions each worth 20 points for a total of 100 points. You may **not** use any notes, books or calculators. Ask for clarification if you are not sure what a question is asking you to do.

1.
(i) (4 pts) Give a careful statement of the Chain Rule.

Solution:
If the function g is differentiable at c, and the function f is differentiable at $g(c)$, then the composite function $f \circ g$ is differentiable at c and

$$(f \circ g)'(c) = f'(g(c)).g'(c)$$

(ii) (6pts) Use the Chain Rule to find the derivative of $\cos\sqrt{x^2+1}$.

Solution:

$$\cos\sqrt{x^2+1} = f \circ g(x)$$

where $f(x) = \cos x$ and $g(x) = \sqrt{x^2+1}$.

$$f'(x) = \sin x, \quad g'(x) = \frac{2x}{2\sqrt{x^2+1}}$$

where we use the Chain Rule again to calculate $g'(x)$. Hence

$$\cos' \sqrt{x^2+1} = \frac{-x\sin x}{\sqrt{x^2+1}}.$$

Find the following limits:
(iii) (5pts)

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Solution:

$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ is a limit of the type zero divided by zero. Multiply top and bottom by a radical designed to simplify the top.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{2}{1+1} = 1$$

(iv) (5pts)

$$\lim_{x\to\infty} \frac{6e^x - 5x^{10}e^{-x}}{e^x + 14xe^{-x}}.$$

Solution:

$$\lim_{x\to\infty} \frac{6e^x - 5x^{10}e^{-x}}{e^x + 14xe^{-x}} = \lim_{x\to\infty} \frac{6 - 5x^{10}e^{-2x}}{1 + 14xe^{-2x}} = \frac{6}{1} = 6$$

since $\lim_{x\to\infty} x^n e^{-mx} = 0$ for any positive integers m and n.

2. (i) (5 pts) Write down the equation of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.

Solution:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

(ii) (10 pts) Use implicit differentiation to find $\frac{dy}{dx}$ for the implicitly defined curve

$$y^2(2 - x) + xy = x^3 + 1$$

Solution:

$$\begin{aligned}
4y\frac{dy}{dx} - 2xy\frac{dy}{dx} - y^2 + x\frac{dy}{dx} + y &= 3x^2 \\
\frac{dy}{dx}(4y - 2xy + x) &= 3x^2 + y^2 - y \\
\frac{dy}{dx} &= \frac{3x^2 + y^2 - y}{4y - 2xy + x}.
\end{aligned}$$

(iii) (5pts) Find the equation of the tangent line to the curve from part (ii) at the point $(1, 1)$.

Solution:
From part (ii) putting in $x = 1$, $y = 1$ we get

$$\frac{dy}{dx} = \frac{3+1-1}{4-2+1} = \frac{3}{3} = 1.$$

Hence equation of the tangent line is

$$y - 1 = 1(x - 1)$$

or

$$y = x.$$

3. (i) (4 pts) State carefully the definition of the derivative $f'(x)$ of a function f at x in terms of a limit.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(ii) (8 pts) Explain what geometric quantity the quotient that appears in the definition of the derivative represents. Draw a figure to illustrate your answer. What geometric quantity does $f'(x)$ represent? Draw this on your figure also.

Solution:

$\frac{f(x+h)-f(x)}{h}$ is the slope of the secant line, i.e. the line through the points $(x, f(x))$ and $(x+h, f(x+h))$.

$f'(x)$ represents the slope of the tangent line to $y = f(x)$ at the point $(x, f(x))$.

(iii) (8 pts) Compute the derivative of the function

$$f(x) = \frac{1}{x} \quad x \neq 0$$

directly from the definition of the derivative as a limit (i.e. you cannot just use the general power law).

Solution:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= -\lim_{h \to 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}.$$

4. (i) (5 pts) Prove that

$$W(t) = W_0 e^{-\lambda t}, \qquad t \geq 0$$

satisfies the differential equation

$$\frac{dW}{dt} = -\lambda W(t)$$

(where both W_0 and λ are positive numbers).

Solution:

$$\frac{dW}{dt} = -\lambda e^{-\lambda t} W_0 = -\lambda W(t).$$

(ii) (7 pts) Suppose $W(t)$ describes the amount of a radioactive material remaining after time t. What quantity does W_0 represent physically? What is the physical interpretation of the constant λ? How does increasing the value of λ affect how fast the material decays? What would happen if λ were a negative number?

Solution:

W_0 is the amount of material initially present (at time $t = 0$).
λ represents the rate at which the material decays.
Increasing λ gives faster decay.
Negative λ would imply exponential growth instead of decay.

(iii) (8 pts) Suppose that $W(0) = 100$ and $W(10) = 50$. Find W_0 and λ in this case (you can leave your answer for λ with a log in it).

Solution:

$$W(0) = 100, \quad \Rightarrow W_0 e^0 = W_0 = 100$$
$$W(10) = 50, \quad \Rightarrow 100e^{-10\lambda} = 50, \Rightarrow e^{-10\lambda} = \frac{1}{2}$$
$$\Rightarrow -10\lambda = \log\frac{1}{2} = \log 1 - \log 2 = \log 2$$
$$\Rightarrow \lambda = \frac{\log 2}{10}.$$

5. (i) (3pts) Define what it means for a function f to be continuous at $x = c$. Give a precise definition.

Solution:

$$\lim_{x \to c} f(x) = f(c)$$

(ii) (3pts) What does the Sandwich Theorem say?

Solution:
If $f(x) \leq g(x) \leq h(x)$ holds for all x near c, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$$

then

$$\lim_{x \to c} g(x) \quad \text{exists and also equals } l.$$

(iii) (5pts) Use the Sandwich Theorem to find $\lim_{x \to 0} f(x)$ for the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution:
For any $x \neq 0$, we have $-1 \leq \sin \frac{1}{x} \leq 1$.
Hence for any $x \geq 0$, we have $-x \leq x \sin \frac{1}{x} \leq x$
and for any $x \leq 0$, we have $-x \geq x \sin \frac{1}{x} \geq x$
The Sandwich Theorem then tells us that

$$\lim_{x \to 0+} x \sin \frac{1}{x} = 0 = \lim_{x \to 0-} x \sin \frac{1}{x}$$

So $\lim_{x \to 0} x \sin \frac{1}{x}$ exists and equals 0.

(iv) (2pts) Is the function from part (iii) continuous at $x = 0$? Explain your answer.

Solution:

Part (iii) shows that $\lim_{x \to 0} f(x) = 0$. Since $f(0) = 0$, then by the definition in part (i) f is continuous at 0.

(v) (7pts) Use the Intermediate Value Theorem to prove that

$$x^3 - 2x + 3 = 0$$

has a solution in $(-3, -1)$.

Solution:

Let $f(x) = x^3 - 2x + 3$. $f(x)$ is a polynomial, so it is continuous for all real numbers x.

$f(-3) = (-3)^3 - 2.(-3) + 3 = -27 + 6 + 3 = -18.$
$f(-1) = (-1)^3 - 2.(-1) + 3 = -1 + 2 + 3 = 4.$

Since $f(-3) < 0 < f(-1)$ and f is continuous, the Intermediate Value Theorem implies that there is some $x \in (-3, -1)$ such that $f(x) = 0$.