NAME:

SECTION:

SSN:

April 14, 2003

IN-CLASS MIDTERM II

Calculus 106 (Biological and Social Sciences) – Professor Haskins

Attempt all parts of all questions.
Remember to show all your work and to write clearly, neatly.
There are 5 questions each worth 20 points for a total of 100 points. You may **not** use any notes, books or calculators. Ask for clarification if you are not sure what a question is asking you to do.

1.
(i) (5 pts) Give a careful statement of L'Hospital's Rule.

Solution: Suppose that f and g are differentiable functions and that either

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

(ii) (5pts) Find

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right).$$

Solution: This is a limit of the form "$\infty - \infty$" which we must rewrite to try to use L'Hospital's Rule.

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0} \frac{\sin x - x}{x \sin x}.$$

This is now a limit in the form "$0/0$". Applying L'Hospital's Rule once gives

$$\lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0} \frac{\cos x - 1}{\sin x + x \cos x}$$

This is another "$0/0$" type limit, so applying L'Hospital again gives us

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0.$$

(iii) (5pts) Find

$$\lim_{x\to\infty} \frac{x^{15}+6x^3-1}{\pi x^{15}-3x+7}.$$

Solution: This is a rational function (ratio of polynomials) and there is no need to use L'Hospital. Just divide top and bottom by the highest power of x in this case.

$$\lim_{x\to\infty} \frac{x^{15}+6x^3-1}{\pi x^{15}-3x+7} = \lim_{x\to\infty} \frac{1+6x^{-12}-x^{-15}}{\pi-3x^{-14}+7x^{-14}} = \frac{1+0+0}{\pi+0+0} = \frac{1}{\pi}.$$

(iv) (5pts) Find

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^t dt.$$

Solution: Let $f(x) = \int_0^x e^t dt$, and $g(x) = x$. Then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

Since e^x is continuous, by the Fundamental Theorem of Calculus f is differentiable with $f'(x) = e^x$. Hence by L'Hospital's Rule we have

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^t dt = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1.$$

2. Let

$$f(x) = \frac{2x^2 - 5}{x + 2}, \quad x \neq -2.$$

(i) (5 pts) Determine where f is increasing and where it is decreasing? Find all the local extrema (if any exist).

Solution: f is increasing on intervals where f' is positive and decreasing on intervals where f' is negative.

$$f'(x) = \frac{(x+2)4x - (2x^2-5)1}{(x+2)^2} = \frac{4x^2 + 8x - 2x^2 + 5}{(x+2)^2} = \frac{2x^2 + 8x + 5}{(x+2)^2}$$

$f'(x) = 0 \iff 2x^2 + 8x + 5 = 0$. Solving using the formula for quadratic equations gives us

$$x_{\pm} = \frac{-8 \pm \sqrt{64 - 4.2.5}}{4} = -2 \pm \sqrt{\frac{3}{2}}.$$

Since the bottom of $f'(x)$ is always nonnegative, $f'(x)$ is positive wherever the top is positive, and negative wherever the top is negative. Since $2x^2 + 8x + 5$ is concave up, then it is positive for $x < x_-$ and for $x > x_+$, it is negative for $x \in (x_-, x_+)$.

Hence f is decreasing on (x_-, x_+) and increasing on $(-\infty, x_-)$ and on (x_+, ∞). x_- and x_+ are the only local extrema of f.

(ii) (5 pts) Determine where f is concave up and where it is concave down. Find all the inflection points (if any exist).

Solution:

$$f''(x) = \frac{(x+2)^2.(2x+8) - 2.(x+2).(2x^2+8x+5)}{(x+2)^4}$$

$$= 2\frac{(x+2)(x+4) - 2x^2 - 8x - 5}{(x+2)^3} = 2\frac{x^2 + 6x + 8 - 2x^2 - 8x - 5}{(x+2)^3}$$

$$= -2\frac{x^2 + 2x - 3}{(x+2)^3} = \frac{-2(x+3)(x-1)}{(x+2)^3}.$$

From this we see that $f''(x)$ changes sign at $x = -3$ and $x = 1$ and hence these are inflection points. Also the sign of $f''(x)$ changes as x changes from less than -2 to greater than -2.

f is concave up when $f''(x) > 0$ i.e. for $x \in (-\infty, -3)$ and $(-2, 1)$.

f is concave down when $f''(x) < 0$ i.e for $x \in (-3, -2)$ and $(1, \infty)$.

(iii) (5pts) Find all asymptotes of f (horizontal, vertical or oblique).

Solution: f is a rational function where the degree of the top is one greater than the degree of the bottom. Hence f will have oblique asymptotes as $x \to \pm\infty$ which we can find as follows:

$$\frac{2x^2-5}{x+2} = \frac{2x(x+2)-4x-5}{x+2} = 2x - \frac{4x+5}{x+2} = 2x - \frac{4(x+2)-8+5}{x+2} = 2x - 4 + \frac{3}{x+2}.$$

Hence the oblique asymptote is the straight line $y = 2x - 4$.

(iv) (5pts) Using the information from the previous three parts give a careful sketch of the graph of f.

3. (i) (20 pts) Find the smallest possible perimeter for a rectangle whose area is 25 square feet.

Solution: Let h and w denote the height and width of the rectangle in feet. Then the perimeter is given by $2h+2w$, and the area is given by $h*w$. Since $hw = 25$, we have $w = \frac{25}{h}$. Hence we can rewrite the perimeter entirely in terms of h by

$$P(h) = 2h + \frac{50}{h}.$$

To find the minimum of P we first find the local extrema of P by finding h such that $P'(h) = 0$.

$$P'(h) = 2 - \frac{50}{h^2}.$$

Hence

$$P'(h) = 0 \iff \frac{50}{h^2} = 2 \iff h^2 = 25 \iff h = \pm 5.$$

Since height is a nonnegative quantity we must have $h = 5$. Since $P''(h) = \frac{100}{h^3}$, $P''(5) > 0$, and hence by the 2nd derivative test $h = 5$ is a local minimum of P.

The only thing left is to check the values of P at the ends of the domain. In this case this means finding the limit of P as $h \to 0$ and as $h \to \infty$. In both cases the limit of P is $+\infty$. Hence the global minimum of P occurs at the local min $h = 5$ where the value is $P(5) = 2*5 + 2*5 = 20$.

4. (i) (5 pts) Give a careful statement of the Mean Value Theorem. Give precise geometric interpretations of the quantities which occur in the theorem. Draw a picture to illustrate.

Solution: Suppose f is continuous on $[a, b]$ and differentiable on (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some} \ \ c \in (a, b).$$

The left hand side is the slope of the secant line through the 2 points $P_1 = (a, f(a))$ and $P_2 = (b, f(b))$. The right hand side is the slope of the tangent line to the graph of f at the point c. Geometrically, the MVT says that for at least one point between a and b the tangent line at that point is parallel to the secant line between P_1 and P_2.

(ii) (5 pts) Suppose f is continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = 0$ for all $x \in (a, b)$. Use the Mean Value Theorem to prove that f is constant on $[a, b]$.

Solution: Let $x < y$ be any two points in $[a, b]$. Then applying the MVT to the interval $[x, y]$ we get that

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ for some } \ c \in (x, y).$$

But since $f'(c) = 0$ for any $c \in (a, b)$, this says the slope of the secant line through x and y is zero. Hence $f(x) = f(y)$. Since x and y are any two points in $[a, b]$ this tells us that f is constant.

(iii) (5 pts) What two properties define the inverse $f^{-1}(x)$ of a function $f(x)$? Give a formula for the derivative of $f^{-1}(x)$ in terms of the derivative of f.

Solution: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ are the two defining properties of the inverse function.

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

(iv) (5pts) Use part (iii) to find the derivative of $\ln x$, assuming that the derivative of e^x is once again e^x.

Solution: $\ln x$ is the inverse function of e^x. Hence

$$\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

using the fact that $e^{\ln x} = x$.

5. (i) (5pts) Find the definite integral

$$\int_0^2 te^{t^2} dt.$$

Solution: Since

$$\frac{d}{dt}\frac{1}{2}e^{t^2} = te^{t^2}$$

$$\int_0^2 te^{t^2} dt = \int_0^2 \frac{d}{dt}\frac{1}{2}e^{t^2} dt = \frac{1}{2}e^{t^2}\Big|_0^2 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1).$$

(ii) (10pts) Find the value of $a \in [0, 2\pi]$ that maximizes

$$\int_0^a \sin x dx.$$

Explain your reasoning carefully.

Solution: Define a function $f(a) = \int_0^a \sin x dx$ on $[0, 2\pi]$. Since $\sin x$ is continuous, by the Fundamental Theorem of Calculus we have

$$\frac{df}{da} = \sin a.$$

Since sin is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$, this implies that $f(a)$ is increasing on $(0, \pi)$ and decreasing on $(\pi, 2\pi)$. Hence the maximum value of f occurs when $a = \pi$. (in this case $f(\pi) = -\cos a|_0^\pi = 2$.)

(iii) (5pts) Suppose that

$$\int_0^x f(t)dt = \frac{1}{2}\tan 2x.$$

Find $f(x)$.

Solution: We can differentiate the LHS of the above equation using the Fundamental Theorem of Calculus and the RHS using the Chain Rule. Setting these two sides equal gives us

$$f(x) = \sec^2 2x.$$