$= \frac{(-1)^n n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)\,n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered

Ratio Test to find the radius of convergence R.

$$\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$

$$= \frac{1}{3}|x-4|\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}|x-4|$$

$\frac{1}{3}|x-4| < 1 \quad\Leftrightarrow\quad |x-4| < 3$, so $R = 3$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2\sin 2x$	0
3	$-2^3\cos 2x$	-2^3
4	$2^4\sin 2x$	0
⋮	⋮	⋮

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x.$$

So $R = \infty$ (Ratio Test).

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
⋮	⋮	⋮

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right] = \lim_{n\to\infty}\frac{5|x|}{n+1}$$

$$= 0 < 1 \text{ for all } x$$

So $R = \infty$.

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
⋮	⋮	⋮

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!}\cdot\frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty}\frac{|x|}{n}$$

$$= 0 < 1 \text{ for all } x$$

So $R = \infty$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
⋮	⋮	⋮

$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \ge 1$, so

$$\ln x = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1 \text{ for convergence,}$$

so $|x-2| < 2 \quad\Rightarrow\quad R = 2$.

$$= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}, \text{ so } \int\frac{\sin x}{x}\,dx = \int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

20. $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n n!}{3^n(n+1)\,n!}(x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(n+1)}(x-4)^n$. Now

$f(5) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(n+1)} = \sum_{n=0}^{\infty}(-1)^n b_n$ is the sum of an alternating series that satisfies (i) $b_{n+1} \le b_n$ and

(ii) $\lim_{n\to\infty} b_n = 0$, so by the Alternating Series Estimation Theorem, $|R_5(5)| = |f(5) - T_5(5)| \le b_6$, and

$b_6 = \frac{1}{3^6(7)} = \frac{1}{5103} \approx 0.000196 < 0.0002$; that is, the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

$$\ldots - \frac{q}{\ldots - d)^2} = \frac{q}{D^2} - \frac{q}{D^2(1+d/D)^2} = \frac{q}{D^2}\left[1 - \left(1 + \frac{d}{D}\right)^{-2}\right].$$

... Binomial Series to expand $(1 + d/D)^{-2}$:

$$\ldots - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right] = \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right]$$

$$\ldots \right) = 2qd \cdot \frac{1}{D^3}$$

... larger than d; that is, when P is far away from the dipole.

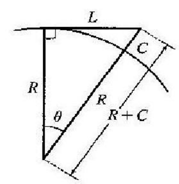
26. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \Leftrightarrow v \approx \sqrt{gL/(2\pi)}$.

(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$, and so $v^2 \approx \frac{gL}{2\pi} \cdot \frac{2\pi d}{L} \Leftrightarrow v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x\ \tanh x$	0
3	$2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$ is less than the first neglected term, which is $\frac{|f'''(0)|}{3!}\left(\frac{2\pi d}{L}\right)^3 = \frac{1}{3}\left(\frac{2\pi d}{L}\right)^3$. If $L > 10d$, then $\frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 < \frac{1}{3}\left(2\pi \cdot \frac{1}{10}\right)^3 = \frac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\frac{gL}{2\pi} \cdot \frac{\pi^3}{375} \approx 0.0132gL$.

27. (a) L is the length of the arc subtended by the angle θ, so $L = R\theta \Rightarrow \theta = L/R$. Now $\sec\theta = (R + C)/R \Rightarrow R\sec\theta = R + C \Rightarrow C = R\sec\theta - R = R\sec(L/R) - R$.



(b) If $f(x) = \sec x$, then $f'(x) = \sec x\ \tan x$,

$f''(x) = \sec^3 x + \sec x\ \tan^2 x$, $f'''(x) = 5\sec^3 x\ \tan x + \sec x\ \tan^3 x$,

$f^{(4)}(x) = 5\sec^5 x + 18\sec^3 x\ \tan^2 x + \sec x\ \tan^4 x$. So $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$,

$f^{(4)}(0) = 5$, and $\sec x \approx T_4(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. By part (a),

$$C \approx R\left[1 + \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4\right] - R = R + \frac{1}{2}R\cdot\frac{L^2}{R^2} + \frac{5}{24}R\cdot\frac{L^4}{R^4} - R = \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

(c) Taking $L = 100$ km and $R = 6370$ km, the formula in part (a) says that

$C = R\sec(L/R) - R = 6370\sec(100/6370) - 6370 \approx 0.785\,009\,965\,44$ km. The formula in part (b) says that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3} = \frac{100^2}{2\cdot 6370} + \frac{5\cdot 100^4}{24\cdot 6370^3} \approx 0.785\,009\,957\,36 \text{ km}.$$

The difference between these two results is only 0.000 000 008 08 km, or 0.000 008 08 m!