hence, $A = 2$. [Alternatively, to find the coefficients A and B, we may use substitution as follows: substitute 2 for x in $(\star)$ to get $-7 = 7B \Leftrightarrow B = -1$, then substitute -5 for x in $(\star)$ to get $-14 = -7A \Leftrightarrow A = 2$.]

Thus, $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx = \int\left(\frac{2}{x+5} + \frac{-1}{x-2}\right)dx = 2\ln|x+5| - \ln|x-2| + C.$

To find the constants in problems involving partial fractions, we may use the coefficient comparison method or the substitution method (as in the solution for Exercise 9) or a combination of both methods.

10. $\dfrac{1}{(t+4)(t-1)} = \dfrac{A}{t+4} + \dfrac{B}{t-1} \Rightarrow 1 = A(t-1) + B(t+4)$.

$t = 1 \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$. $t = -4 \Rightarrow 1 = -5A \Rightarrow A = -\frac{1}{5}$. Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int\left(\frac{-1/5}{t+4} + \frac{1/5}{t-1}\right)dt = -\tfrac{1}{5}\ln|t+4| + \tfrac{1}{5}\ln|t-1| + C \quad\text{or}\quad \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right| + C$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1) + B(x+1)$. Substituting 1 for x gives $1 = 2B \Leftrightarrow B = \frac{1}{2}$. Substituting -1 for x gives $1 = -2A \Leftrightarrow A = -\frac{1}{2}$. Thus,

$$\begin{aligned}\int_2^3 \frac{1}{x^2-1}\,dx &= \int_2^3\left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1|\right]_2^3 \\ &= \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \left(-\tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]\end{aligned}$$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x - 1 = A(x+2) + B(x+1)$. Substituting -2 for x gives $-3 = -B \Leftrightarrow B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\begin{aligned}\int_0^1 \frac{x-1}{x^2+3x+2}\,dx &= \int_0^1\left(\frac{-2}{x+1} + \frac{3}{x+2}\right)dx = \left[-2\ln|x+1| + 3\ln|x+2|\right]_0^1 \\ &= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]\end{aligned}$$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C.$

15. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \Rightarrow 2x + 3 = A(x+1) + B$. Take $x = -1$ to get $B = 1$, and equate coefficients of x to get $A = 2$. Now

$$\begin{aligned}\int_0^1 \frac{2x+3}{(x+1)^2}\,dx &= \int_0^1\left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]dx = \left[2\ln(x+1) - \frac{1}{x+1}\right]_0^1 \\ &= 2\ln 2 - \tfrac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \tfrac{1}{2}\end{aligned}$$

16. $\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)}=\dfrac{A}{x-3}+\dfrac{B}{x+2}$. Then $3x-4=A(x+2)+B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \Leftrightarrow A=1$ and $-10=-5B \Leftrightarrow B=2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx=\int_0^1\left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx=\left[\frac{1}{2}x^2+x+\ln|x-3|+2\ln(x+2)\right]_0^1$$
$$=\left(\tfrac{1}{2}+1+\ln 2+2\ln 3\right)-(0+0+\ln 3+2\ln 2)=\tfrac{3}{2}+\ln 3-\ln 2=\tfrac{3}{2}+\ln\tfrac{3}{2}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)}=\dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3} \Rightarrow 4y^2-7y-12=A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy=\int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy=\left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$=2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2$$
$$=2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{9}{5}(3\ln 2-\ln 3)=\tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{x^2+2x-1}{x^3-x}=\dfrac{x^2+2x-1}{x(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$. Substituting 0 for x gives $-1=-A \Leftrightarrow A=1$. Substituting -1 for x gives $-2=2B \Leftrightarrow B=-1$. Substituting 1 for x gives $2=2C \Leftrightarrow C=1$. Thus,

$$\int\frac{x^2+2x-1}{x^3-x}\,dx=\int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right)dx=\ln|x|-\ln|x+1|+\ln|x-1|+C=\ln\left|\frac{x(x-1)}{x+1}\right|+C.$$

19. $\dfrac{1}{(x+5)^2(x-1)}=\dfrac{A}{x+5}+\dfrac{B}{(x+5)^2}+\dfrac{C}{x-1} \Rightarrow 1=A(x+5)(x-1)+B(x-1)+C(x+5)^2$.

Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1=A(3)(-3)+B(-3)+C(3^2)=-9A-3B+9C=-9A+\frac{1}{2}+\frac{1}{4}=-9A+\frac{3}{4}$, so $9A=-\frac{1}{4}$ and $A=-\frac{1}{36}$.

Now $\displaystyle\int\frac{1}{(x+5)^2(x-1)}\,dx=\int\left[\frac{-1/36}{x+5}-\frac{1/6}{(x+5)^2}+\frac{1/36}{x-1}\right]dx=-\frac{1}{36}\ln|x+5|+\frac{1}{6(x+5)}+\frac{1}{36}\ln|x-1|+C.$

0. $\dfrac{x^2}{(x-3)(x+2)^2}=\dfrac{A}{x-3}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2} \Rightarrow x^2=A(x+2)^2+B(x-3)(x+2)+C(x-3)$.

Setting $x=3$ gives $A=\frac{9}{25}$. Take $x=-2$ to get $C=-\frac{4}{5}$, and equate the coefficients of x^2 to get $1=A+B \Rightarrow B=\frac{16}{25}$. Then

$$\int\frac{x^2}{(x-3)(x+2)^2}dx=\int\left[\frac{9/25}{x-3}+\frac{16/25}{x+2}-\frac{4/5}{(x+2)^2}\right]dx=\frac{9}{25}\ln|x-3|+\frac{16}{25}\ln|x+2|+\frac{4}{5(x+2)}+C.$$

$$\int \frac{1}{x^3-1}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx$$

$$= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{3}\int \frac{(3/2)dx}{(x+1/2)^2+3/4}$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K$$

30. $\dfrac{x^3}{x^3+1} = \dfrac{(x^3+1)-1}{x^3+1} = 1 - \dfrac{1}{x^3+1} = 1 - \left(\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}\right) \Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1).$

Equate the terms of degree 2, 1 and 0 to get $0 = A+B$, $0 = -A+B+C$, $1 = A+C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\int \frac{x^3}{x^3+1}\,dx = \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x-\frac{2}{3}}{x^2-x+1}\right]dx = x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln(x^2-x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K$$

31. $\dfrac{1}{x^4-x^2} = \dfrac{1}{x^2(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get

$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x = 1$ gives $C = \frac{1}{2}$,

taking $x = -1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x = 0$

yields $B = -1$. Now $\displaystyle\int \frac{dx}{x^4-x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right]dx = \frac{1}{x} + \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

32. Let $u = x^4 + 5x^2 + 4 \Rightarrow du = (4x^3 + 10x)\,dx = 2(2x^3+5x)\,dx$, so

$$\int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_4^{10}\frac{du}{u} = \tfrac{1}{2}\big[\ln|u|\big]_4^{10} = \tfrac{1}{2}(\ln 10 - \ln 4) = \tfrac{1}{2}\ln\tfrac{5}{2}.$$

33. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{[(x+1)^2+3]^2}\,dx = \int \frac{u-4}{(u^2+3)^2}\,du$ [with $u = x+1$]

$$= \int \frac{u\,du}{(u^2+3)^2} - 4\int \frac{du}{(u^2+3)^2} = \frac{1}{2}\int \frac{dv}{v^2} - 4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \quad \left[\begin{array}{l} v = u^2+3 \text{ in the first integral;} \\ u = \sqrt{3}\tan\theta \text{ in the second} \end{array}\right]$$

$$= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}(x+1)}{x^2+2x+4}\right] + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) - \frac{2(x+1)}{3(x^2+2x+4)} + C$$

34. $\dfrac{x^4+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} \;\Rightarrow\; x^4+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$

Setting $x=0$ gives $A=1$, and equating the coefficients of x^4 gives $1=A+B$, so $B=0$. Now

$$\frac{C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{x^4+1}{x(x^2+1)^2} - \frac{1}{x} = \frac{1}{x}\left[\frac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2}\right] = \frac{-2x}{(x^2+1)^2},$$

so we can take $C=0$, $D=-2$, and $E=0$. Hence,

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int\left[\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right]dx = \ln|x| + \frac{1}{x^2+1} + C$$

35. Let $u=\sqrt{x}$, so $u^2=x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}\,2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4\left(1+\frac{4}{u^2-4}\right)du \quad \text{[by long division]}$$
$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)} \quad (\star)$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(\star)$ is

$$2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du = 2+8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4 = 2+\Big[2\ln|u-2| - 2\ln|u+2|\Big]_3^4$$
$$= 2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 = 2+2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2+2\ln\tfrac{2/6}{1/5}$$
$$= 2+2\ln\tfrac{5}{3} \text{ or } 2+\ln\left(\tfrac{5}{3}\right)^2 = 2+\ln\tfrac{25}{9}$$

36. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx = 3u^2\,du \;\Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1\left(3u-3+\frac{3}{1+u}\right)du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

37. Let $u=\sqrt[3]{x^2+1}$. Then $x^2 = u^3-1$, $2x\,dx = 3u^2\,du \;\Rightarrow$

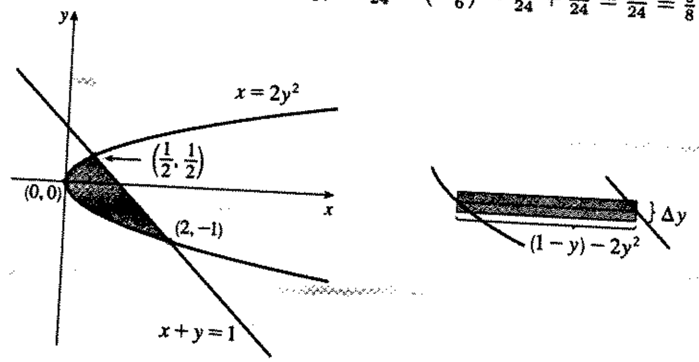
$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \tfrac{3}{2}\int (u^4-u)\,du = \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C.$$

38. Let $u=\sqrt{x}$. Then $x=u^2$, $dx = 2u\,du \;\Rightarrow$

$$\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}.$$

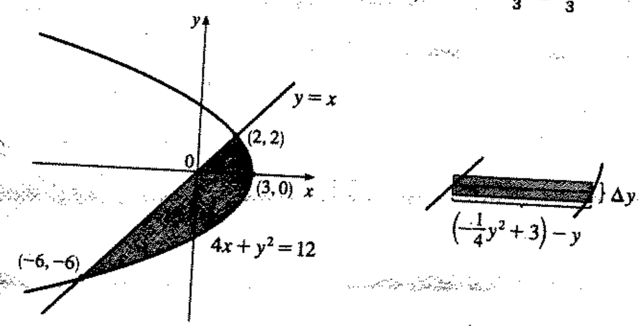
11. $2y^2 = 1 - y \;\Leftrightarrow\; 2y^2 + y - 1 = 0 \;\Leftrightarrow\; (2y-1)(y+1) = 0 \;\Leftrightarrow\; y = \frac{1}{2}$ or -1, so $x = \frac{1}{2}$ or 2 and

$$A = \int_{-1}^{1/2} \left[(1-y) - 2y^2\right] dy = \int_{-1}^{1/2} (1 - y - 2y^2)\, dy = \left[y - \tfrac{1}{2}y^2 - \tfrac{2}{3}y^3\right]_{-1}^{1/2}$$
$$= \left(\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{12}\right) - \left(-1 - \tfrac{1}{2} + \tfrac{2}{3}\right) = \tfrac{7}{24} - \left(-\tfrac{5}{6}\right) = \tfrac{7}{24} + \tfrac{20}{24} = \tfrac{27}{24} = \tfrac{9}{8}$$



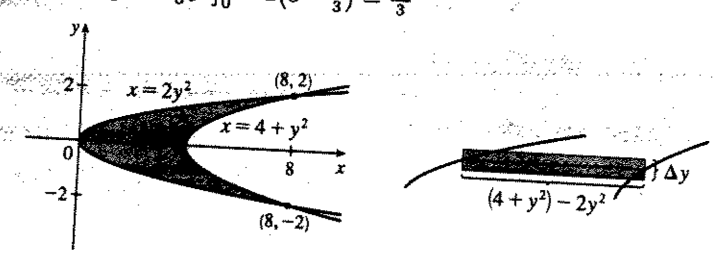
12. $4x + x^2 = 12 \;\Leftrightarrow\; (x+6)(x-2) = 0 \;\Leftrightarrow\; x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy = \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2}$$
$$= \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18) = 22 - \tfrac{2}{3} = \tfrac{64}{3}$$



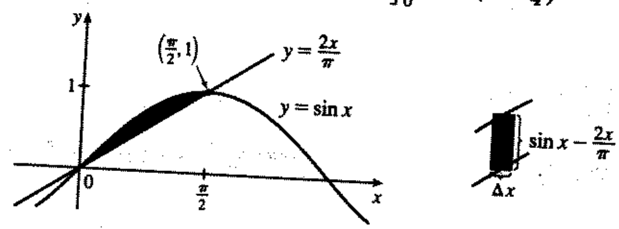
13. $2y^2 = 4 + y^2 \;\Leftrightarrow\; y^2 = 4 \;\Leftrightarrow\; y = \pm 2$, so

$$A = \int_{-2}^{2} \left[(4 + y^2) - 2y^2\right] dy = 2\int_{0}^{2} (4 - y^2)\, dy \qquad \text{[by symmetry]}$$
$$= 2\left[4y - \tfrac{1}{3}y^3\right]_0^2 = 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3}$$



14. By observation, $y = \sin x$ and $y = 2x/\pi$ intersect at $(0, 0)$ and $(\pi/2, 1)$ for $x \geq 0$.

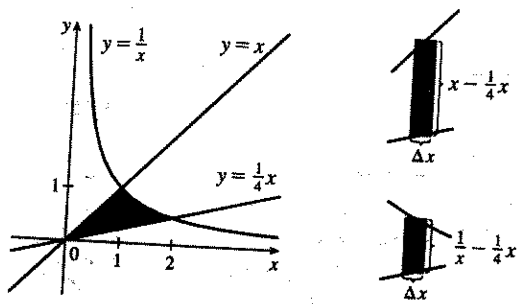
$$A = \int_0^{\pi/2} \left(\sin x - \frac{2x}{\pi}\right) dx = \left[-\cos x - \frac{1}{\pi}x^2\right]_0^{\pi/2} = \left(0 - \frac{\pi}{4}\right) - (-1) = 1 - \frac{\pi}{4}$$



15. $1/x = x \;\Leftrightarrow\; 1 = x^2 \;\Leftrightarrow\; x = \pm 1$ and $1/x = \frac{1}{4}x \;\Leftrightarrow\; 4 = x^2 \;\Leftrightarrow\; x = \pm 2$, so for $x > 0$,

$$A = \int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx = \int_0^1 \left(\frac{3}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

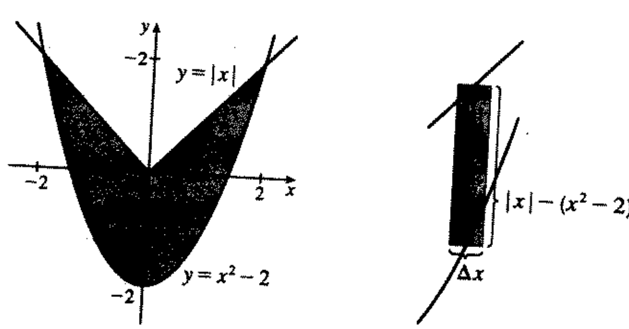
$$= \left[\frac{3}{8}x^2\right]_0^1 + \left[\ln|x| - \frac{1}{8}x^2\right]_1^2 = \tfrac{3}{8} + \left(\ln 2 - \tfrac{1}{2}\right) - \left(0 - \tfrac{1}{8}\right) = \ln 2$$



16. For $x > 0$, $x = x^2 - 2 \;\Rightarrow\; 0 = x^2 - x - 2 \;\Rightarrow\; 0 = (x - 2)(x + 1) \;\Rightarrow\; x = 2$. By symmetry,

$$\int_{-2}^{2} \left[|x| - (x^2 - 2)\right] dx = 2\int_0^2 \left[x - (x^2 - 2)\right] dx = 2\int_0^2 (x - x^2 + 2)\, dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x\right]_0^2$$

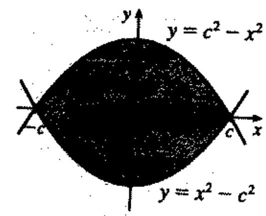
$$= 2\left(2 - \tfrac{8}{3} + 4\right) = \tfrac{20}{3}$$



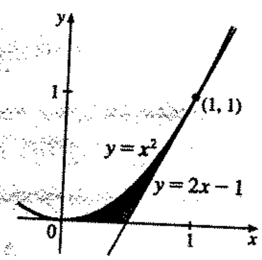
31. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

$A = 4\int_0^c (c^2 - x^2)\,dx = 4\left[c^2x - \frac{1}{3}x^3\right]_0^c = 4\left(c^3 - \frac{1}{3}c^3\right) = 4\left(\frac{2}{3}c^3\right) = \frac{8}{3}c^3.$

So $A = 576 \quad\Leftrightarrow\quad \frac{8}{3}c^3 = 576 \quad\Leftrightarrow\quad c^3 = 216 \quad\Leftrightarrow\quad c = \sqrt[3]{216} = 6$. Note that $c = -6$ is another solution, since the graphs are the same.



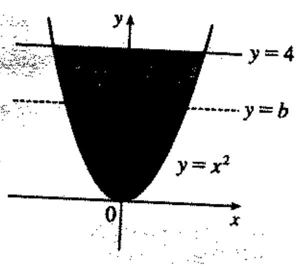
32.



We start by finding the equation of the tangent line to $y = x^2$ at the point $(1, 1)$: $y' = 2x$, so the slope of the tangent is $2(1) = 2$, and its equation is $y - 1 = 2(x - 1)$, or $y = 2x - 1$. We would need two integrals to integrate with respect to x, but only one to integrate with respect to y.

$$A = \int_0^1 \left[\tfrac{1}{2}(y+1) - \sqrt{y}\right] dy = \left[\tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\right]_0^1 = \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3} = \tfrac{1}{12}$$

33.



By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \quad\Rightarrow\quad x = \sqrt{y}$. We are looking for a number b such that

$\int_0^b \sqrt{y}\,dy = \int_b^4 \sqrt{y}\,dy \quad\Rightarrow\quad \frac{2}{3}\left[y^{3/2}\right]_0^b = \frac{2}{3}\left[y^{3/2}\right]_b^4 \quad\Rightarrow$

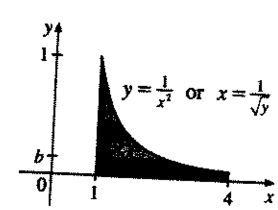
$b^{3/2} = 4^{3/2} - b^{3/2} \quad\Rightarrow\quad 2b^{3/2} = 8 \quad\Rightarrow\quad b^{3/2} = 4 \quad\Rightarrow\quad b = 4^{2/3} \approx 2.52.$

34. (a) We want to choose a so that

$$\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \quad\Rightarrow\quad \left[\frac{-1}{x}\right]_1^a = \left[\frac{-1}{x}\right]_a^4 \quad\Rightarrow\quad -\frac{1}{a} + 1 = -\frac{1}{4} + \frac{1}{a} \quad\Rightarrow\quad \frac{5}{4} = \frac{2}{a} \quad\Rightarrow\quad a = \frac{8}{5}.$$

(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)]. Now the line $y = b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$. This implies that $\int_b^1 (1/\sqrt{y} - 1)\,dy = \frac{1}{2}\cdot\frac{3}{4} \quad\Rightarrow$ $\left[2\sqrt{y} - y\right]_b^1 = \frac{3}{8} \quad\Rightarrow\quad 1 - 2\sqrt{b} + b = \frac{3}{8} \quad\Rightarrow\quad b - 2\sqrt{b} + \frac{5}{8} = 0.$

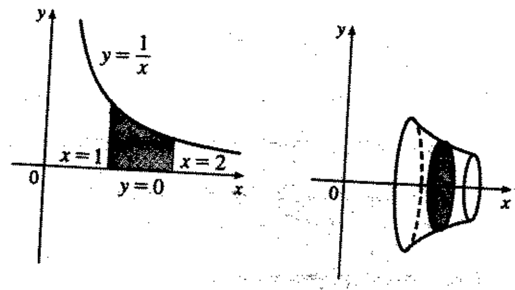
Letting $c = \sqrt{b}$, we get $c^2 - 2c + \frac{5}{8} = 0 \quad\Rightarrow\quad 8c^2 - 16c + 5 = 0$. Thus, $c = \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}$. But $c = \sqrt{b} < 1 \quad\Rightarrow\quad c = 1 - \frac{\sqrt{6}}{4} \quad\Rightarrow$ $b = c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8}(11 - 4\sqrt{6}) \approx 0.1503.$



7.2 Volumes

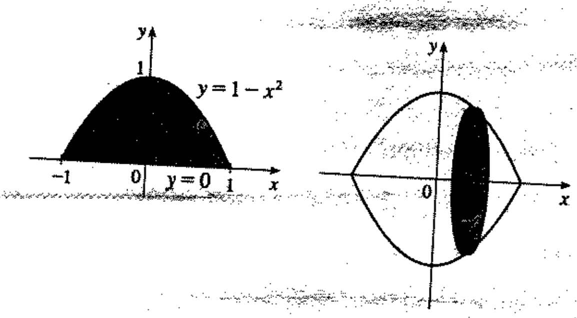
1. A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx$$
$$= \pi \int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2$$
$$= \pi\left[-\tfrac{1}{2} - (-1)\right] = \frac{\pi}{2}$$



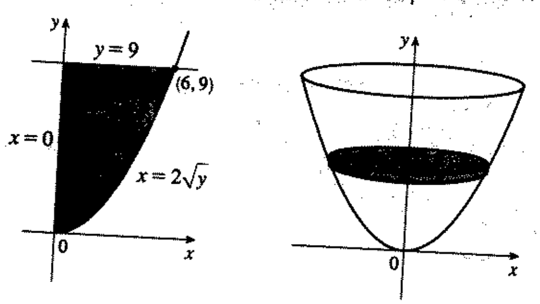
2. A cross-section is a disk with radius $1 - x^2$, so its area is $A(x) = \pi(1 - x^2)^2$.

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(1 - x^2)^2\,dx$$
$$= 2\pi \int_0^1 (1 - 2x^2 + x^4)\,dx = 2\pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1$$
$$= 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 2\pi\left(\tfrac{8}{15}\right) = \tfrac{16\pi}{15}$$



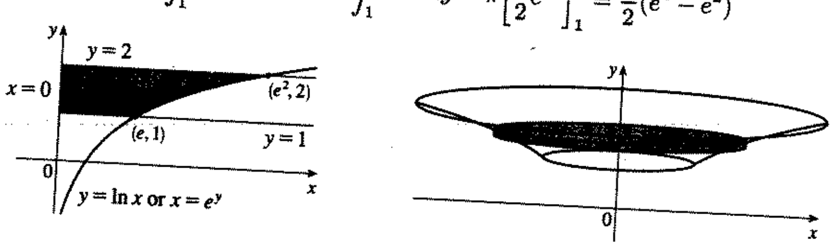
3. A cross-section is a disk with radius $2\sqrt{y}$, so its area is $A(y) = \pi\left(2\sqrt{y}\right)^2$.

$$V = \int_0^9 A(y)\,dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy$$
$$= 4\pi \int_0^9 y\,dy = 4\pi\left[\tfrac{1}{2}y^2\right]_0^9$$
$$= 2\pi(81) = 162\pi$$



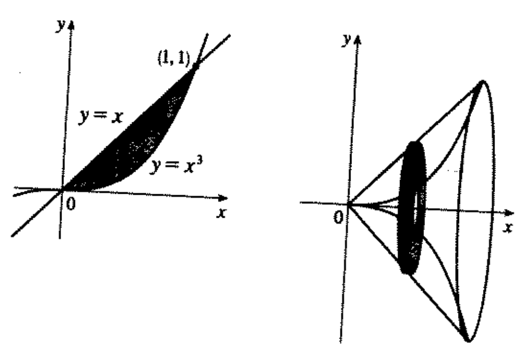
4. A cross-section is a disk with radius e^y, so its area is $A(y) = \pi(e^y)^2$.

$$V = \int_1^2 \pi(e^y)^2\,dy = \pi\int_1^2 e^{2y}\,dy = \pi\left[\frac{1}{2}e^{2y}\right]_1^2 = \frac{\pi}{2}(e^4 - e^2)$$



5. A cross-section is a washer (annulus) with inner radius x^3 and outer radius x, so its area is $A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx$$
$$= \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{7}\right) = \tfrac{4\pi}{21}$$

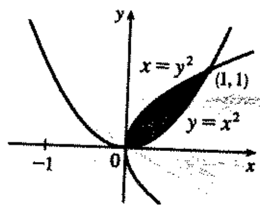


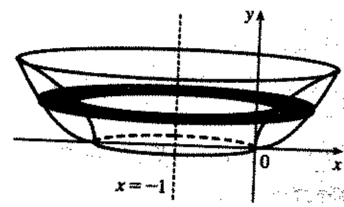
11. $y = x^2 \Rightarrow x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi \left\{ \left[\sqrt{y} - (-1)\right]^2 - \left[y^2 - (-1)\right]^2 \right\} dy = \pi \int_0^1 \left[\left(\sqrt{y} + 1\right)^2 - \left(y^2 + 1\right)^2 \right] dy$$

$$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1 \right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2 \right) dy$$

$$= \pi \left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 \right]_0^1 = \pi\left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3}\right) = \tfrac{29}{30}\pi$$

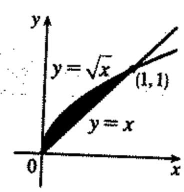


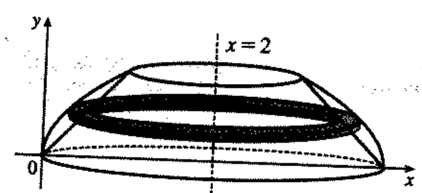


12. $y = \sqrt{x} \Rightarrow x = y^2$, so the outer radius is $2 - y^2$.

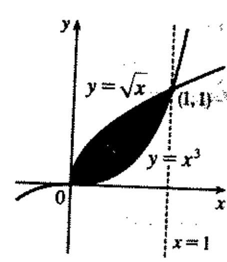
$$V = \int_0^1 \pi[(2 - y^2)^2 - (2 - y)^2]\, dy = \pi \int_0^1 [(4 - 4y^2 + y^4) - (4 - 4y + y^2)]\, dy$$

$$= \pi \int_0^1 (y^4 - 5y^2 + 4y)\, dy = \pi\left[\tfrac{1}{5}y^5 - \tfrac{5}{3}y^3 + 2y^2\right]_0^1 = \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right) = \tfrac{8}{15}\pi$$





13. $y = \sqrt{x} \Rightarrow x = y^2$ and $y = x^3 \Rightarrow x = \sqrt[3]{y}$. A cross-section is a washer with inner radius $1 - \sqrt[3]{y}$ and outer radius $1 - y^2$, so its area is $A(y) = \pi(1 - y^2)^2 - \pi(1 - \sqrt[3]{y})^2$.



$$V = \int_0^1 A(y)\, dy = \int_0^1 \left[\pi(1 - y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy$$

$$= \pi \int_0^1 \left[(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})\right] dy$$

$$= \pi \int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3})\, dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13\pi}{30}$$

14. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x^3$, so its area is $A(x) = \pi(1 - x^3)^2 - \pi\left(1 - \sqrt{x}\right)^2$.

$$V = \int_0^1 A(x)\, dx = \int_0^1 \left[\pi(1 - x^3)^2 - \pi\left(1 - \sqrt{x}\right)^2\right] dx = \pi \int_0^1 [(1 - 2x^3 + x^6) - (1 - 2x^{1/2} + x)]\, dx$$

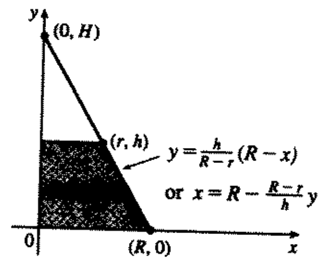
$$= \pi \int_0^1 (-2x^3 + x^6 + 2x^{1/2} - x)\, dx = \pi\left[-\tfrac{1}{2}x^4 + \tfrac{1}{7}x^7 + \tfrac{4}{3}x^{3/2} - \tfrac{1}{2}x^2\right]_0^1 = \pi\left(-\tfrac{1}{2} + \tfrac{1}{7} + \tfrac{4}{3} - \tfrac{1}{2}\right) = \tfrac{10\pi}{21}$$

26. $V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy = \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$

$= \pi \left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$

$= \pi\left[R^2 h - R(R-r)h + \frac{1}{3}(R-r)^2 h\right]$

$= \frac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \frac{1}{3}\pi h(R^2 + Rr + r^2)$



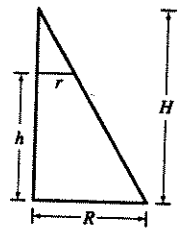
Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

$Hr = HR - hR \Rightarrow hR = H(R-r) \Rightarrow H = \frac{hR}{R-r}$. Now

$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 (H-h)$ [by Exercise 25]

$= \frac{1}{3}\pi R^2 \frac{hR}{R-r} - \frac{1}{3}\pi r^2 \frac{rh}{R-r}$ $\left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)}\right]$

$= \frac{1}{3}\pi h \frac{R^3 - r^3}{R-r} = \frac{1}{3}\pi h(R^2 + Rr + r^2) = \frac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]h = \frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$



where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 28 for a related result.)

27. $x^2 + y^2 = r^2 \Leftrightarrow x^2 = r^2 - y^2$

$V = \pi \int_{r-h}^{r} (r^2 - y^2)\, dy = \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r}$

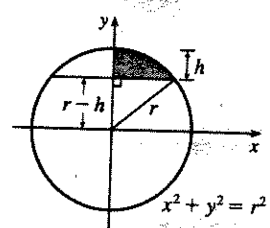
$= \pi\left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$

$= \pi\left\{\frac{2}{3}r^3 - \frac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right]\right\}$

$= \frac{1}{3}\pi\left\{2r^3 - (r-h)\left[3r^2 - (r^2 - 2rh + h^2)\right]\right\}$

$= \frac{1}{3}\pi\left\{2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right]\right\} = \frac{1}{3}\pi(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3)$

$= \frac{1}{3}\pi(3rh^2 - h^3) = \frac{1}{3}\pi h^2(3r - h)$, or, equivalently, $\pi h^2\left(r - \frac{h}{3}\right)$



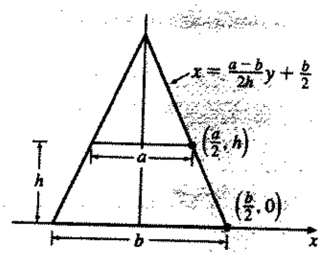
28. An equation of the line is $x = \frac{\Delta x}{\Delta y}y + (x\text{-intercept}) = \frac{a/2 - b/2}{h - 0}y + \frac{b}{2} = \frac{a-b}{2h}y + \frac{b}{2}$.

$V = \int_0^h A(y)\, dy = \int_0^h (2x)^2\, dy = \int_0^h \left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right]^2 dy$

$= \int_0^h \left[\frac{a-b}{h}y + b\right]^2 dy = \int_0^h \left[\frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2\right] dy$

$= \left[\frac{(a-b)^2}{3h^2}y^3 + \frac{b(a-b)}{h}y^2 + b^2 y\right]_0^h = \frac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h$

$= \frac{1}{3}(a^2 - 2ab + b^2 + 3ab)h = \frac{1}{3}(a^2 + ab + b^2)h$



[Note that this can be written as $\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$, as in Exercise 26.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\frac{1}{3}b^2 h$.