34. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\, d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta = \int_0^{2\pi} \sqrt{5e^{4\theta}}\, d\theta$

$= \sqrt{5} \int_0^{2\pi} e^{2\theta}\, d\theta = \frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$

35. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2\left(\theta^2 + 4\right)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \frac{1}{3}\left[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}\right] = \frac{8}{3}\left[(\pi^2+1)^{3/2} -\right.$

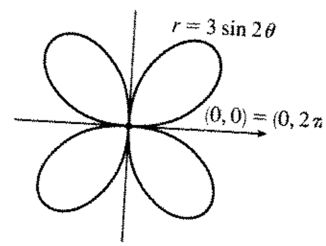
36. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1}\, d\theta \overset{21}{=} \left[\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_0^{2\pi}$

$= \pi\sqrt{4\pi^2+1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$

37. The curve $r = 3\sin 2\theta$ is completely traced with $0 \le \theta \le 2\pi$.

$r^2 + (dr/d\theta)^2 = (3\sin 2\theta)^2 + (6\cos 2\theta)^2 \Rightarrow$

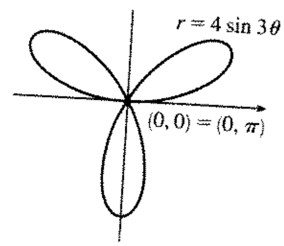
$L = \int_0^{2\pi} \sqrt{9\sin^2 2\theta + 36\cos^2 2\theta}\, d\theta \approx 29.0653$



38. The curve $r = 4\sin 3\theta$ is completely traced with $0 \le \theta \le \pi$.

$r^2 + (dr/d\theta)^2 = (4\sin 3\theta)^2 + (12\cos 3\theta)^2 \Rightarrow$

$L = \int_0^{\pi} \sqrt{16\sin^2 3\theta + 144\cos^2 3\theta}\, d\theta \approx 26.7298$



9.5 Conic Sections in Polar Coordinates

1. The directrix $y = 6$ is above the focus at the origin, so we use the form with "$+ e\sin\theta$" in the denominator. (See Theo

and Figure 8.) $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{\frac{7}{4}\cdot 6}{1 + \frac{7}{4}\sin\theta} = \dfrac{42}{4 + 7\sin\theta}$

2. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with "$+ e\cos\theta$" in the denominator. $e =$

a parabola, so an equation is $r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{1\cdot 4}{1 + 1\cos\theta} = \dfrac{4}{1 + \cos\theta}$

3. The directrix $x = -5$ is to the left of the focus at the origin, so we use the form with "$- e\cos\theta$" in the denominator.

$r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{\frac{3}{4}\cdot 5}{1 - \frac{3}{4}\cos\theta} = \dfrac{15}{4 - 3\cos\theta}$

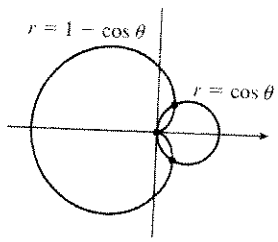
4. The directrix $y = -2$ is below the focus at the origin, so we use the form with "$- e\sin\theta$" in the denominator.

$r = \dfrac{ed}{1 - e\sin\theta} = \dfrac{2\cdot 2}{1 - 2\sin\theta} = \dfrac{4}{1 - 2\sin\theta}$

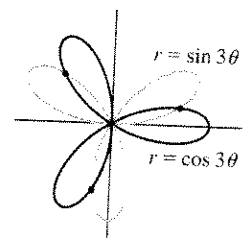
relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we continue:

$$\begin{aligned} A &= 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} \\ &= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) \\ &= 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \approx 204.16\ \text{m}^2 \end{aligned}$$

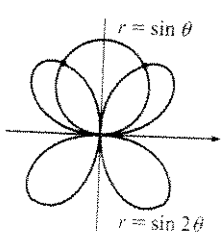
29. The curves intersect at the pole since $\left(0, \frac{\pi}{2}\right)$ satisfies $r = \cos\theta$ and $(0, 0)$ satisfies $r = 1 - \cos\theta$. Now $\cos\theta = 1 - \cos\theta \;\Rightarrow\; 2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3} \;\Rightarrow$ the other intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.



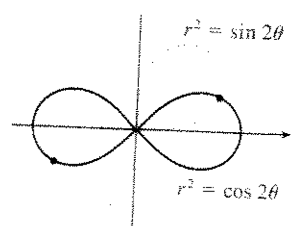
30. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \;\Rightarrow\; \tan 3\theta = 1 \;\Rightarrow\; 3\theta = \frac{\pi}{4} + n\pi$ [n any integer] $\Rightarrow\; \theta = \frac{\pi}{12} + \frac{\pi}{3}n \;\Rightarrow\; \theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta \;\Leftrightarrow\; \sin\theta\,(1 - 2\cos\theta) = 0 \;\Leftrightarrow\; \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = 0, \pi, \frac{\pi}{3}, -\frac{\pi}{3} \;\Rightarrow\; \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ (by symmetry) are the other intersection points.



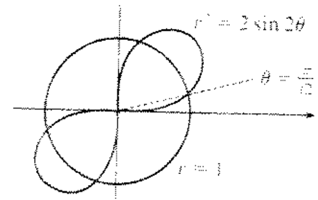
32. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \;\Rightarrow\; \tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\Rightarrow\; \theta = \frac{\pi}{8} + n\pi \;\Rightarrow\; \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{\pi/3}\sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\,d\theta = \int_0^{\pi/3}\sqrt{9(\sin^2\theta + \cos^2\theta)}\,d\theta \\ &= 3\int_0^{\pi/3} d\theta = 3[\theta]_0^{\pi/3} = 3\left(\tfrac{\pi}{3}\right) = \pi. \end{aligned}$$

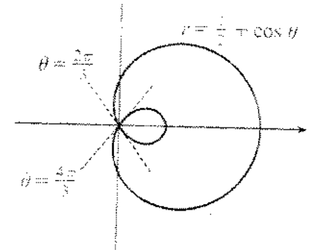
s a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right) = 3\pi$, and since $\theta = 0$ to $\pi = \frac{\pi}{3}$ traces out $\frac{1}{3}$ of the rcle (from $\theta = 0$ to $\theta = \pi$), $\frac{1}{3}(3\pi) = \pi$.

26. $2\sin 2\theta = 1^2 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$.



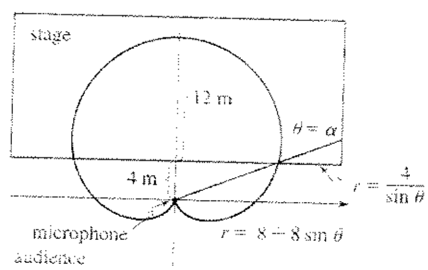
$$A = 4\left[\int_0^{\pi/12} \tfrac{1}{2}\cdot 2\sin 2\theta\, d\theta + \int_{\pi/12}^{\pi/4} \tfrac{1}{2}(1^2)\, d\theta\right]$$
$$= \left[-2\cos 2\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{\pi/4}$$
$$= -2\left(\tfrac{\sqrt{3}}{2} - 1\right) + 2\left(\tfrac{1}{4}\pi - \tfrac{1}{12}\pi\right) = 2 - \sqrt{3} + \tfrac{\pi}{3}$$

27. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.



$$A = 2\left[\int_0^{2\pi/3} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta\right]$$
$$= \int_0^{2\pi/3} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta - \int_{2\pi/3}^{\pi} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta$$
$$= \int_0^{2\pi/3} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta - \int_{2\pi/3}^{\pi} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$
$$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) = \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)$$

28. We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \Leftrightarrow r\sin\theta = 4 \Rightarrow r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \Rightarrow$



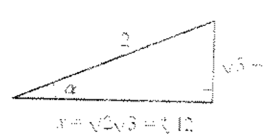
$8\sin\theta + 8\sin^2\theta = 4 \Rightarrow 2\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow$

$\sin\theta = \dfrac{-2 \pm \sqrt{4 + 8}}{4} = \dfrac{-2 \pm 2\sqrt{3}}{4} = \dfrac{-1 + \sqrt{3}}{2}$ [the other value is less than -1] $\Rightarrow \theta = \sin^{-1}\left(\dfrac{\sqrt{3} - 1}{2}\right)$.

This angle is about 21.5° and is denoted by α in the figure.

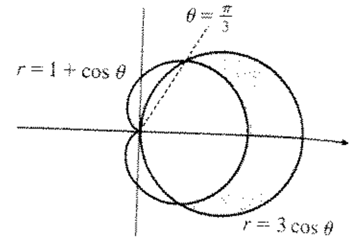
$$A = 2\int_\alpha^{\pi/2} \tfrac{1}{2}(8 + 8\sin\theta)^2 d\theta - 2\int_\alpha^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2 d\theta = 64\int_\alpha^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\, d\theta - 16\int_\alpha^{\pi/2} \csc^2\theta\, d\theta$$
$$= 64\int_\alpha^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta + 16\int_\alpha^{\pi/2}(-\csc^2\theta)\, d\theta$$
$$= 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_\alpha^{\pi/2} + 16\left[\cot\theta\right]_\alpha^{\pi/2} = 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_\alpha^{\pi/2}$$
$$= 16\left[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)\right] = 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

From the figure, $x^2 + (\sqrt{3} - 1)^2 = 2^2 \Rightarrow x^2 = 4 - (3 - 2\sqrt{3} + 1)$ $\Rightarrow x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric



21. $3\cos\theta = 1 + \cos\theta \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.

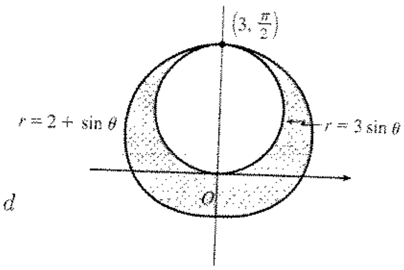
$$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$
$$= \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta$$
$$= \int_0^{\pi/3} [4(1+\cos 2\theta) - 2\cos\theta - 1]\, d\theta$$
$$= \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$



22. To find the shaded area A, we'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi\left(\frac{3}{2}\right)^2$ since $r = 3\sin\theta$ is a circle with radius $\frac{3}{2}$.

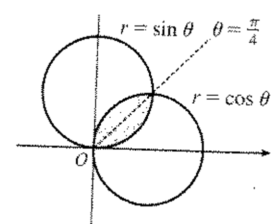
$$A_1 = \int_0^{2\pi} \tfrac{1}{2}(2+\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\, d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left[4 + 4\sin\theta + \tfrac{1}{2}\cdot(1-\cos 2\theta)\right] d\theta = \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos 2\theta\right) d$$
$$= \tfrac{1}{2}\left[\tfrac{9}{2}\theta - 4\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}\left[(9\pi - 4) - (-4)\right] = \tfrac{9\pi}{2}$$

So $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$.



23. $A = 2\int_0^{\pi/4} \frac{1}{2}\sin^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta$

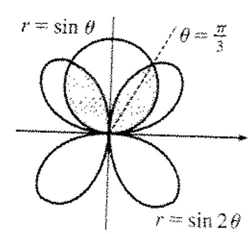
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\cdot 1\right) - (0 - 0)\right]$$
$$= \tfrac{1}{8}\pi - \tfrac{1}{4}$$



24. $r = \sin 2\theta$ takes on both positive and negative values.

$\sin\theta = \pm\sin 2\theta = \pm 2\sin\theta\,\cos\theta \quad\Rightarrow\quad \sin\theta\,(1 \pm 2\cos\theta) = 0$.

From the figure we can see that the intersections occur where $\cos\theta = \pm\frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

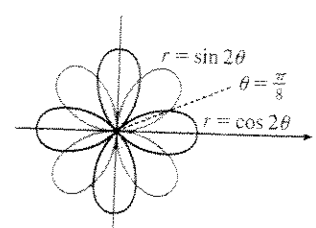


$$A = 2\left[\int_0^{\pi/3} \tfrac{1}{2}\sin^2\theta\, d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}\sin^2 2\theta\, d\theta\right] = \int_0^{\pi/3} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}(1 - \cos 4\theta)\, d\theta$$
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{1}{2}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{\pi/3}^{\pi/2} = \tfrac{4\pi - 3\sqrt{3}}{16}$$

25. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow$

$2\theta = \frac{\pi}{4} \quad\Rightarrow\quad \theta = \frac{\pi}{8} \quad\Rightarrow$

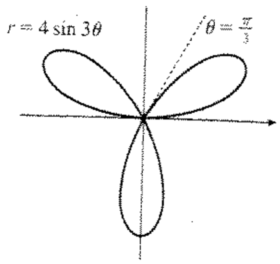
$$A = 8\cdot 2\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\, d\theta = 8\int_0^{\pi/8} \tfrac{1}{2}(1 - \cos 4\theta)\, d$$
$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{1}{2}\pi - 1$$



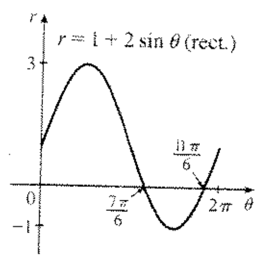
16. $A = \int_0^{\pi/3} \frac{1}{2}(4\sin 3\theta)^2\,d\theta = 8\int_0^{\pi/3} \sin^2 3\theta\,d\theta$

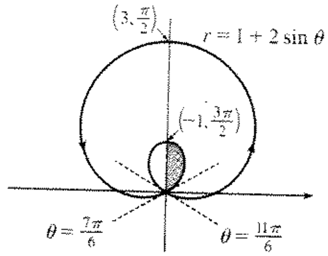
$= 4\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta$

$= 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{4\pi}{3}$



17.



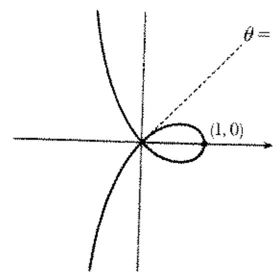


This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$A = 2\int_{7\pi/6}^{3\pi/2} \frac{1}{2}(1+2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}(1+4\sin\theta+4\sin^2\theta)\,d\theta = \int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta$

$= \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2} = \left(\frac{9\pi}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}$

18. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve $r = 0 \Rightarrow 2\cos\theta - \dfrac{1}{\cos\theta} = 0 \Rightarrow$ $2\cos^2\theta - 1 = 0 \Rightarrow \cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow$ $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ for $0 \le \theta \le \pi$ with $\theta \ne \frac{\pi}{2}$.

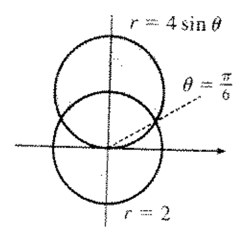


$A = 2\int_0^{\pi/4}\frac{1}{2}(2\cos\theta - \sec\theta)^2\,d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\,d\theta$

$= \int_0^{\pi/4}\left[4\cdot\frac{1}{2}(1+\cos 2\theta) - 4 + \sec^2\theta\right]d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\,d\theta$

$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\frac{\pi}{2} + 1 + 1\right) - 0 = 2 - \frac{\pi}{2}$

19. $4\sin\theta = 2 \Leftrightarrow \sin\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (for $0 \le \theta \le 2\pi$). We'll subtract the unshaded area from the shaded area for $\pi/6 \le \theta \le \pi/2$ and double that value.



$A = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}(4\sin\theta)^2\,d\theta - 2\int_{\pi/6}^{\pi/2}\frac{1}{2}(2)^2\,d\theta = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}\left[(4\sin\theta)^2 - 2^2\right]d\theta$

$= \int_{\pi/6}^{\pi/2}(16\sin^2\theta - 4)\,d\theta = \int_{\pi/6}^{\pi/2}\left[8(1-\cos 2\theta) - 4\right]d\theta = \int_{\pi/6}^{\pi/2}(4 - 8\cos 2\theta)\,d\theta$

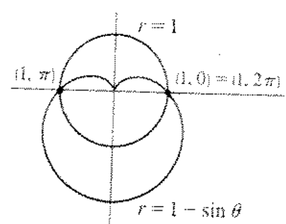
$= \left[4\theta - 4\sin 2\theta\right]_{\pi/6}^{\pi/2} = (2\pi - 0) - \left(\frac{2\pi}{3} - 4\cdot\frac{\sqrt{3}}{2}\right) = \frac{4}{3}\pi + 2\sqrt{3}$

20. $1 - \sin\theta = 1 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or $\pi \Rightarrow$

$A = \int_\pi^{2\pi}\frac{1}{2}\left[(1-\sin\theta)^2 - 1\right]d\theta = \frac{1}{2}\int_\pi^{2\pi}(\sin^2\theta - 2\sin\theta)\,d\theta$

$= \frac{1}{4}\int_\pi^{2\pi}(1 - \cos 2\theta - 4\sin\theta)\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi} = \frac{1}{4}\pi + 2$

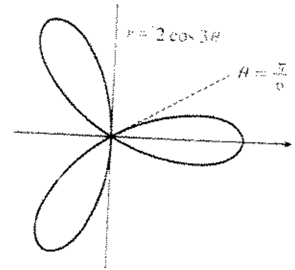


11. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

$A = 6\int_0^{\pi/6} \frac{1}{2}(2\cos 3\theta)^2\,d\theta = 12\int_0^{\pi/6} \cos^2 3\theta\,d\theta$

$= \frac{12}{2}\int_0^{\pi/6}(1 + \cos 6\theta)\,d\theta$

$= 6\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\frac{\pi}{6}\right) = \pi$

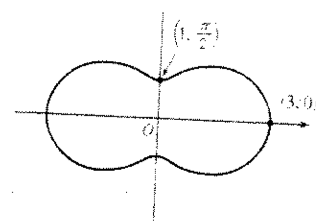


12. $A = \int_0^{2\pi} \frac{1}{2}(2 - \cos 2\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4 - 4\cos 2\theta + \cos^2 2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(4 - 4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\,d\theta$

$= \frac{1}{2}\left[\frac{9}{2}\theta - 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$

$= \frac{1}{2}(9\pi) = \frac{9\pi}{2}$



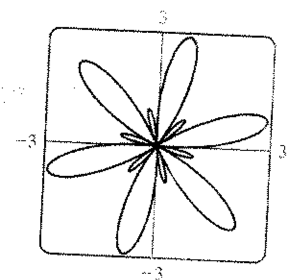
13. $A = \int_0^{2\pi} \frac{1}{2}(1 + 2\sin 6\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 4\sin 6\theta + 4\sin^2 6\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[1 + 4\sin 6\theta + 4 \cdot \frac{1}{2}(1 - \cos 12\theta)\right]\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}(3 + 4\sin 6\theta - 2\cos 12\theta)\,d\theta$

$= \frac{1}{2}\left[3\theta - \frac{2}{3}\cos 6\theta - \frac{1}{6}\sin 12\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[\left(6\pi - \frac{2}{3} - 0\right) - \left(0 - \frac{2}{3} - 0\right)\right] = 3\pi.$



4. $A = \int_0^{\pi} \frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta = 2\int_0^{\pi/2}\frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta$

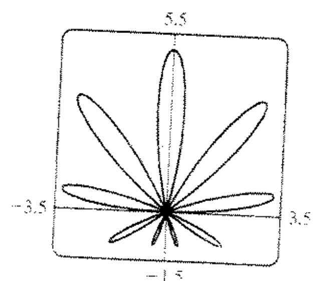
$= \int_0^{\pi/2}(4\sin^2\theta + 12\sin\theta\,\sin 9\theta + 9\sin^2 9\theta)\,d\theta$

$= \int_0^{\pi/2}\left[2(1 - \cos 2\theta) + 12 \cdot \frac{1}{2}(\cos(\theta - 9\theta) - \cos(\theta + 9\theta)) + \frac{9}{2}(1 - \cos 18\theta)\right]\,d\theta$

[integration by parts could be used for $\int \sin\theta\,\sin 9\theta\,d\theta$]

$= \int_0^{\pi/2}\left(2 - 2\cos 2\theta + 6\cos 8\theta - 6\cos 10\theta + \frac{9}{2} - \frac{9}{2}\cos 18\theta\right)\,d\theta$

$= \left[\frac{13}{2}\theta - \sin 2\theta + \frac{3}{4}\sin 8\theta - \frac{3}{5}\sin 10\theta - \frac{1}{4}\sin 18\theta\right]_0^{\pi/2} = \frac{13}{4}\pi$

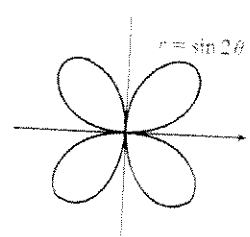


The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

$A = \int_0^{\pi/2}\frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1 - \cos 4\theta)\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$



9.4 Areas and Lengths in Polar Coordinates

1. $r=\sqrt{\theta},\ 0\le\theta\le\frac{\pi}{4}.\quad A=\int_0^{\pi/4}\frac12 r^2\,d\theta=\int_0^{\pi/4}\frac12\left(\sqrt{\theta}\right)^2d\theta=\int_0^{\pi/4}\frac12\theta\,d\theta=\left[\frac14\theta^2\right]_0^{\pi/4}=\frac{1}{64}\pi^2$

2. $r=e^{\theta/2}\ ,\pi\le\theta\le 2\pi.\quad A=\int_\pi^{2\pi}\frac12\left(e^{\theta/2}\right)^2d\theta=\int_\pi^{2\pi}\frac12 e^\theta\,d\theta=\frac12\left[e^\theta\right]_\pi^{2\pi}=\frac12\left(e^{2\pi}-e^\pi\right)$

3. $r=\sin\theta,\ \frac{\pi}{3}\le\theta\le\frac{2\pi}{3}.$

$$A=\int_{\pi/3}^{2\pi/3}\frac12\sin^2\theta\,d\theta=\frac14\int_{\pi/3}^{2\pi/3}(1-\cos 2\theta)\,d\theta=\frac14\left[\theta-\frac12\sin 2\theta\right]_{\pi/3}^{2\pi/3}$$

$$=\frac14\left[\frac{2\pi}{3}-\frac12\sin\frac{4\pi}{3}-\frac{\pi}{3}+\frac12\sin\frac{2\pi}{3}\right]=\frac14\left[\frac{2\pi}{3}-\frac12\left(-\frac{\sqrt3}{2}\right)-\frac{\pi}{3}+\frac12\left(\frac{\sqrt3}{2}\right)\right]=\frac14\left(\frac{\pi}{3}+\frac{\sqrt3}{2}\right)=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$$

4. $r=\sqrt{\sin\theta},\ 0\le\theta\le\pi.\quad A=\int_0^\pi\frac12\left(\sqrt{\sin\theta}\right)^2d\theta=\int_0^\pi\frac12\sin\theta\,d\theta=\left[-\frac12\cos\theta\right]_0^\pi=\frac12+\frac12=1$

5. $r=\theta\ ,0\le\theta\le\pi.\quad A=\int_0^\pi\frac12\theta^2d\theta=\left[\frac16\theta^3\right]_0^\pi=\frac16\pi^3$

6. $r=1+\sin\theta,\ \frac{\pi}{2}\le\theta\le\pi.$

$$A=\int_{\pi/2}^{\pi}\frac12(1+\sin\theta)^2\,d\theta=\frac12\int_{\pi/2}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)d\theta=\frac12\int_{\pi/2}^{\pi}\left[1+2\sin\theta+\frac12(1-\cos 2\theta)\right]d\theta$$

$$=\frac12\left[\theta-2\cos\theta+\frac12\theta-\frac14\sin 2\theta\right]_{\pi/2}^{\pi}=\frac12\left[\pi+2+\frac{\pi}{2}-0-\left(\frac{\pi}{2}-0+\frac{\pi}{4}-0\right)\right]=\frac12\left(\frac{3\pi}{4}+2\right)=\frac{3\pi}{8}+$$

7. $r=4+3\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$

$$A=\int_{-\pi/2}^{\pi/2}\frac12(4+3\sin\theta)^2d\theta=\frac12\int_{-\pi/2}^{\pi/2}\left(16+24\sin\theta+9\sin^2\theta\right)d\theta$$

$$=\frac12\int_{-\pi/2}^{\pi/2}\left(16+9\sin^2\theta\right)d\theta\quad\text{[by Theorem 4.5.6(b)]}$$

$$=\frac12\cdot 2\int_0^{\pi/2}\left[16+9\cdot\frac12(1-\cos 2\theta)\right]d\theta\quad\text{[by Theorem 4.5.6(a)]}$$

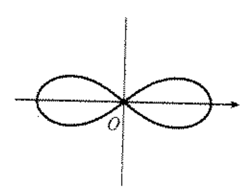
$$=\int_0^{\pi/2}\left(\frac{41}{2}-\frac92\cos 2\theta\right)d\theta=\left[\frac{41}{2}\theta-\frac94\sin 2\theta\right]_0^{\pi/2}=\left(\frac{41\pi}{4}-0\right)-(0-0)=\frac{41\pi}{4}$$

8. $r=\sin 4\theta,\ 0\le\theta\le\frac{\pi}{4}.\quad A=\int_0^{\pi/4}\frac12\sin^2 4\theta\,d\theta=\int_0^{\pi/4}\frac14(1-\cos 8\theta)\,d\theta=\left[\frac14\theta-\frac1{32}\sin 8\theta\right]_0^{\pi/4}=\frac{\pi}{16}$

9. The curve $r^2=4\cos 2\theta$ goes through the pole when $\theta=\pi/4$, so we'll find the area for $0\le\theta\le\pi/4$ and multiply it by 4.

$$A=4\int_0^{\pi/4}\frac12 r^2\,d\theta=2\int_0^{\pi/4}(4\cos 2\theta)\,d\theta=8\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$=4\left[\sin 2\theta\right]_0^{\pi/4}=4(1-0)=4$$

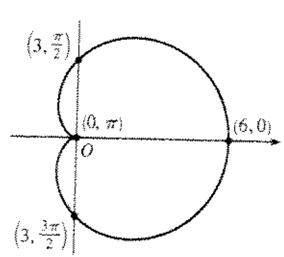


10. $A=\int_0^{2\pi}\frac12 r^2\,d\theta=\int_0^{2\pi}\frac12[3(1+\cos\theta)]^2\,d\theta$

$$=\frac92\int_0^{2\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

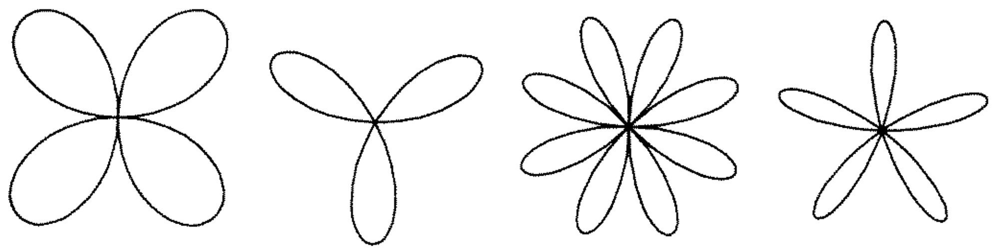
$$=\frac92\int_0^{2\pi}\left[1+2\cos\theta+\frac12(1+\cos 2\theta)\right]d\theta$$

$$=\frac92\left[\frac32\theta+2\sin\theta+\frac14\sin 2\theta\right]_0^{2\pi}=\frac{27}{2}\pi$$



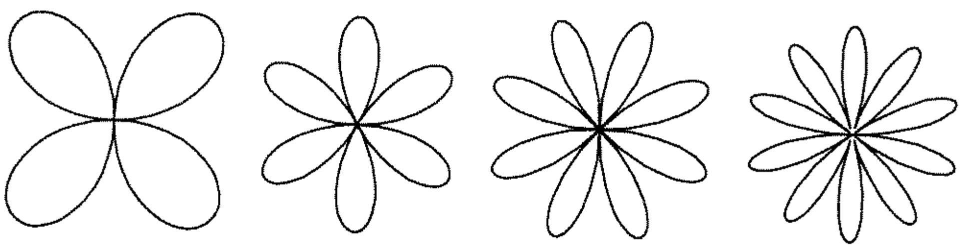
63. **(a)** $r = \sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \cos n\pi + \cos n\theta \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$



$n = 2$ $n = 3$ $n = 4$ $n = 5$

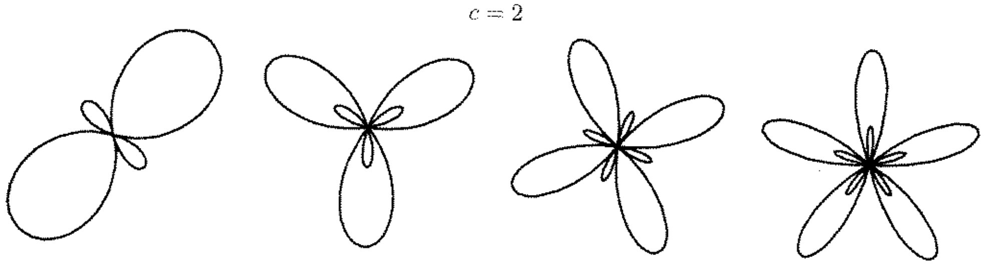
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

64. $r = 1 + c\sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 63: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$c = 2$



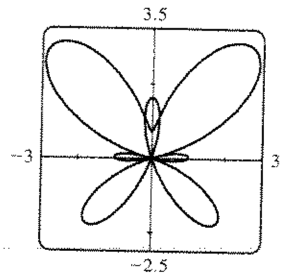
$n = 2$ $n = 3$ $n = 4$ $n = 5$

Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c)\sin n\theta = 1 + c\sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the

Note for Exercises 57–60: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under `Options State`. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 59, $x = r\cos\theta = [2 - 5\sin(\theta/6)]\cos\theta$, $y = r\sin\theta = [2 - 5\sin(\theta/6)]\sin\theta$.

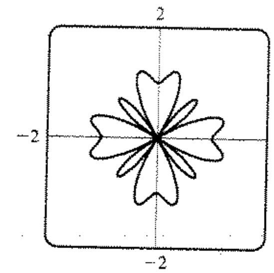
57. $r = e^{\sin\theta} - 2\cos(4\theta)$.

The parameter interval is $[0, 2\pi]$.



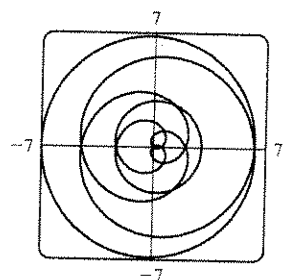
58. $r = \sin^2(4\theta) + \cos(4\theta)$.

The parameter interval is $[0, 2\pi]$.



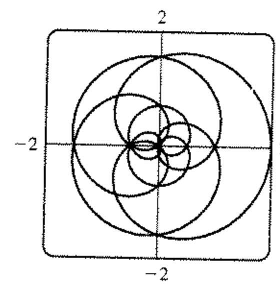
59. $r = 2 - 5\sin(\theta/6)$.

The parameter interval is $[-6\pi, 6\pi]$.

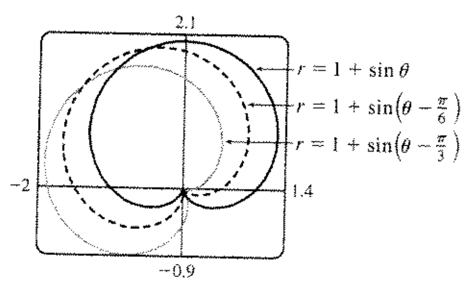


60. $r = \cos(\theta/2) + \cos(\theta/3)$.

The parameter interval is $[-6\pi, 6\pi]$.

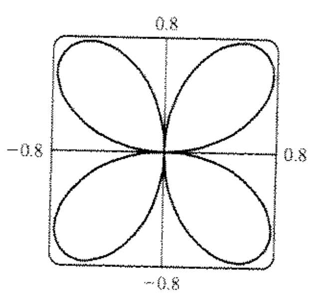


61.



It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.

62.



From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\,\sin 2\theta \quad\Rightarrow$

$$dy/d\theta = 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta$$

$$= 2\sin\theta\,(2\cos^2\theta - 1) + \cos\theta\,(2\sin\theta\,\cos\theta) = 2\sin\theta\,(3\cos^2\theta - 1)$$

In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad\Leftrightarrow\quad \sin\theta = \sqrt{\frac{2}{3}} \quad\Leftrightarrow$

$y = 2\sin^2\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{9} \approx 0.77$.

at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$. $dx/d\theta = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \Rightarrow 2\theta = 0$ or π $\Leftrightarrow$ $\theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $(0, \frac{\pi}{2})$.

52. $dy/d\theta = e^\theta \sin\theta - e^\theta \cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow$ $\theta = -\frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi(n - \frac{1}{4})\right)$.

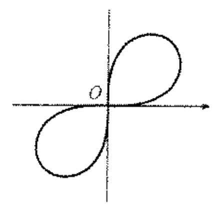
$dx/d\theta = e^\theta \cos\theta - e^\theta \sin\theta = e^\theta(\cos\theta - \sin\theta) = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi(n + \frac{1}{4})\right)$.

53. $r = 1 + \cos\theta \Rightarrow x = r\cos\theta = \cos\theta(1 + \cos\theta)$, $y = r\sin\theta = \sin\theta(1 + \cos\theta) \Rightarrow$ $dy/d\theta = (1 + \cos\theta)\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow$ $\cos\theta = \frac{1}{2}$ or $-1 \Rightarrow \theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \Rightarrow$ horizontal tangent at $(\frac{3}{2}, \frac{\pi}{3})$, $(0, \pi)$ [the pole], and $(\frac{3}{2}, \frac{5\pi}{3})$.

$dx/d\theta = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta(1 + 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \Rightarrow$ $\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \Rightarrow$ vertical tangent at $(2, 0)$, $(\frac{1}{2}, \frac{2\pi}{3})$, and $(\frac{1}{2}, \frac{4\pi}{3})$. Note that the tangent is horizontal, not vertica when $\theta = \pi$, since $\lim\limits_{\theta\to\pi} \dfrac{dy/d\theta}{dx/d\theta} = 0$.

54. By differentiating implicitly, $r^2 = \sin 2\theta \Rightarrow 2r\,(dr/d\theta) = 2\cos 2\theta \Rightarrow$ $dr/d\theta = (1/r)\cos 2\theta$, so

$$\frac{dy}{d\theta} = \frac{1}{r}\cos 2\theta\,\sin\theta + r\cos\theta = \frac{1}{r}(\cos 2\theta\,\sin\theta + r^2\cos\theta)$$
$$= \frac{1}{r}(\cos 2\theta\,\sin\theta + \sin 2\theta\,\cos\theta) = \frac{1}{r}\sin 3\theta$$



This is 0 when $\sin 3\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3}\right)$ and $(0, 0)$. Similarly, $dx/d\theta = (1/r)\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are verti tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6}\right)$ [and $(0, 0)$].

55. $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$ $x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \Rightarrow \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a c with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

56. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r = a\sin\theta$), $dy/d\theta = a\cos\theta\,\sin\theta + a\sin\theta\,\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r = a\cos\theta$), $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

vertical asymptote. Also notice that $x = \sin^2\theta \geq 0$ for all θ, and $x = \sin^2\theta \leq 1$ for all θ. And $x \neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \leq x < 1$.

. (a) $r = \sin(\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π.

(b) $r = \sin(\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π.

(c) $r = \sec(3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}$, and so on.

(d) $r = \theta\sin\theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π. This graph is unbounded, and each time θ moves through an interval of 2π, the same basic shape is repeated (because of the periodic $\sin\theta$ factor) but it gets larger each time (since θ increases each time we go around.)

(e) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.

(f) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

$r = 2\sin\theta \quad\Rightarrow\quad x = r\cos\theta = 2\sin\theta\,\cos\theta = \sin 2\theta,\ y = r\sin\theta = 2\sin^2\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cdot 2\sin\theta\,\cos\theta}{\cos 2\theta\cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta. \text{ When } \theta = \frac{\pi}{6},\ \frac{dy}{dx} = \tan\left(2\cdot\frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}.$$

$r = 2 - \sin\theta \quad\Rightarrow\quad x = r\cos\theta = (2-\sin\theta)\cos\theta,\ y = r\sin\theta = (2-\sin\theta)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2-\sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)} = \frac{2\cos\theta - 2\sin\theta\,\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

$$\text{When } \theta = \frac{\pi}{3},\ \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2}\cdot\frac{2}{2} = \frac{2-\sqrt{3}}{1 - 2\sqrt{3}}.$$

$r = 1/\theta \quad\Rightarrow\quad x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

$$\text{When } \theta = \pi,\ \frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi.$$

$r = \sin 3\theta \quad\Rightarrow\quad x = r\cos\theta = \sin 3\theta\cos\theta,\ y = r\sin\theta = \sin 3\theta\sin\theta \quad\Rightarrow$

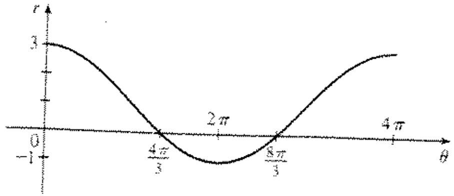
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

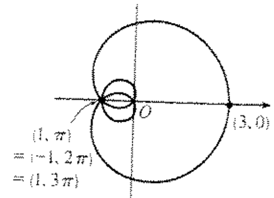
$$\text{When } \theta = \frac{\pi}{6},\ \frac{dy}{dx} = \frac{3(0)(1/2) + 1(\sqrt{3}/2)}{3(0)(\sqrt{3}/2) - 1(1/2)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}.$$

$r = 3\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 3\cos\theta\,\cos\theta,\ y = r\sin\theta = 3\cos\theta\,\sin\theta \quad\Rightarrow$

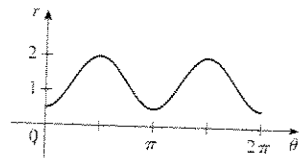
$dy/d\theta = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal

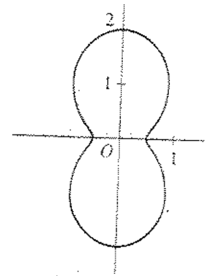
40. $r = 1 + 2\cos(\theta/2)$



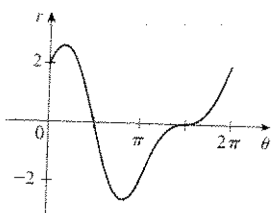


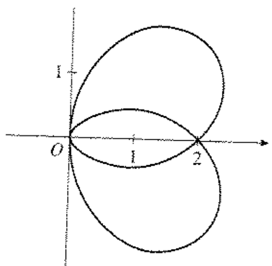
41. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \leq \theta \leq \pi$ and $\pi \leq \theta \leq 2\pi$.



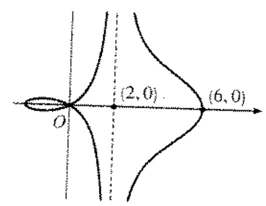


42.

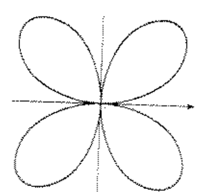




43. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \;\Rightarrow\; (4 + 2\sec\theta) \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ (since we need only consider $0 \leq \theta < 2\pi$), so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \;\Rightarrow\; (4 + 2\sec\theta) \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \;\Rightarrow\; x = 2$ is a vertical asymptote.

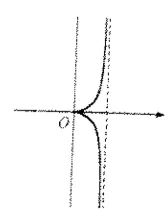


44. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \;\Rightarrow\; r^2 = 4\cos^2\theta\sin^2\theta \;\Rightarrow\; r = \pm 2\cos\theta\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.

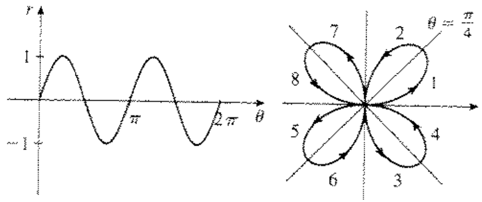


45. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.

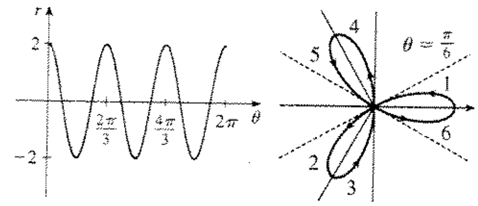
$x = r\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \;\Rightarrow\; \sin\theta\tan\theta \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \;\Rightarrow\; \sin\theta\tan\theta \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \;\Rightarrow\; x = 1$ is a



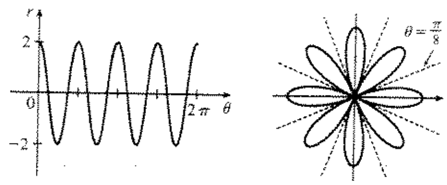
31. $r = \sin 2\theta$



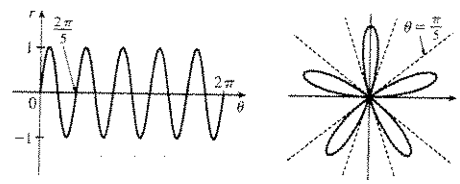
32. $r = 2\cos 3\theta$



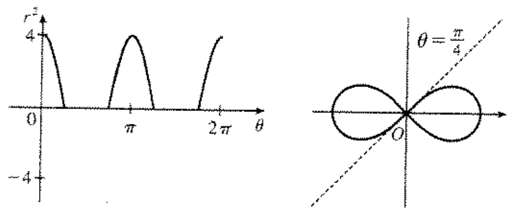
33. $r = 2\cos 4\theta$



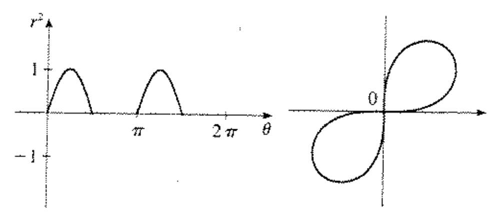
34. $r = \sin 5\theta$



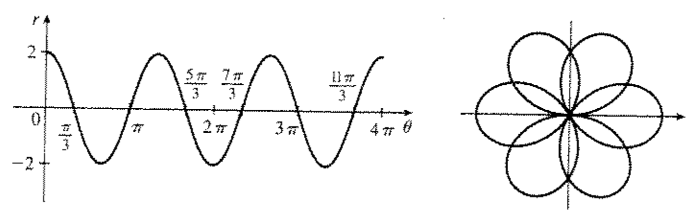
35. $r^2 = 4\cos 2\theta$



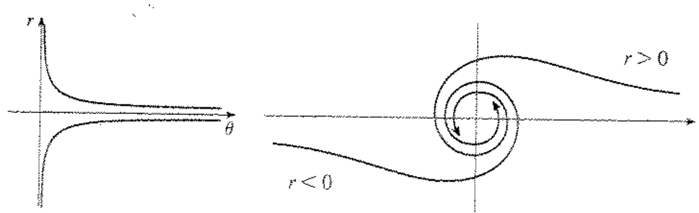
36. $r^2 = \sin 2\theta$



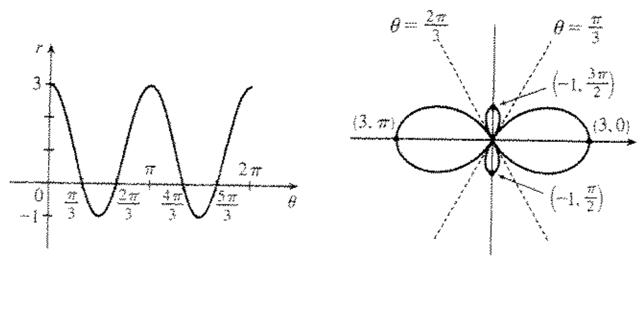
37. $r = 2\cos\left(\frac{3}{2}\theta\right)$



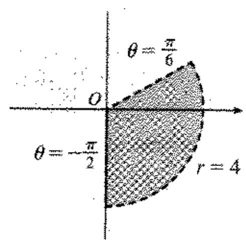
38. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



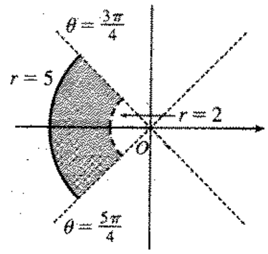
39. $r = 1 + 2\cos 2\theta$



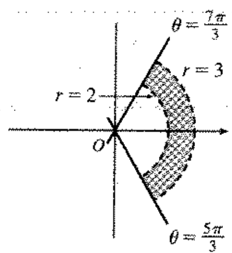
9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r = 4$ nor the line $\theta = \frac{\pi}{6}$.



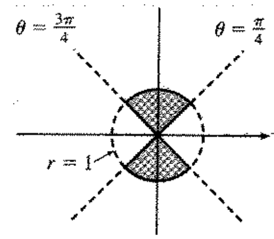
10. $2 < r \le 5, \quad 3\pi/4 < \theta < 5\pi/4$



11. $2 < r < 3, \quad \frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1, \quad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $r = 3\sin\theta \;\Rightarrow\; r^2 = 3r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = 3y \;\Leftrightarrow\; x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2 = 3r\sin\theta \;\Rightarrow\; r(r - 3\sin\theta) = 0 \;\Rightarrow\; r = 0$ or $r = 3\sin\theta$. But $r = 3\sin\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3\sin\theta$ is equivalent to the compound condition ($r = 0$ or $r = 3\sin\theta$).

14. $r = 2\sin\theta + 2\cos\theta \;\Rightarrow\; r^2 = 2r\sin\theta + 2r\cos\theta \;\Leftrightarrow\; x^2 + y^2 = 2y + 2x \;\Leftrightarrow\; (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$ $\Leftrightarrow\; (x-1)^2 + (y-1)^2 = 2$. The first implication is reversible since $r^2 = 2r\sin\theta + 2r\cos\theta \;\Rightarrow\; r = 0$ or $r = 2\sin\theta + 2\cos\theta$, but the curve $r = 2\sin\theta + 2\cos\theta$ passes through the pole ($r = 0$) when $\theta = -\frac{\pi}{4}$, so $r = 2\sin\theta + 2\cos\theta$ includes the single point of $r = 0$. The curve is a circle of radius $\sqrt{2}$, centered at $(1, 1)$.

15. $r = \csc\theta \;\Leftrightarrow\; r = \dfrac{1}{\sin\theta} \;\Leftrightarrow\; r\sin\theta = 1 \;\Leftrightarrow\; y = 1$, a horizontal line 1 unit above the x-axis.

16. $r = \tan\theta\sec\theta = \dfrac{\sin\theta}{\cos^2\theta} \;\Rightarrow\; r\cos^2\theta = \sin\theta \;\Leftrightarrow\; (r\cos\theta)^2 = r\sin\theta \;\Leftrightarrow\; x^2 = y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos\theta = 0$ would imply $\sin\theta = r\cos^2\theta = 0$, contradicting the fact that $\cos^2\theta + \sin^2\theta = 1$.

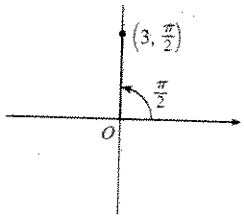
17. $x = -y^2 \;\Leftrightarrow\; r\cos\theta = -r^2\sin^2\theta \;\Leftrightarrow\; \cos\theta = -r\sin^2\theta \;\Leftrightarrow\; r = -\dfrac{\cos\theta}{\sin^2\theta} = -\cot\theta\csc\theta$.

18. $x + y = 9 \;\Leftrightarrow\; r\cos\theta + r\sin\theta = 9 \;\Leftrightarrow\; r = 9/(\cos\theta + \sin\theta)$.

19. $x^2 + y^2 = 2cx \;\Leftrightarrow\; r^2 = 2cr\cos\theta \;\Leftrightarrow\; r^2 - 2cr\cos\theta = 0 \;\Leftrightarrow\; r(r - 2c\cos\theta) = 0 \;\Leftrightarrow\; r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

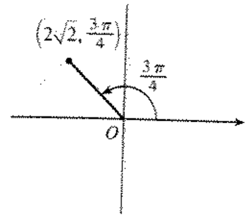
20. $x^2 - y^2 = 1 \;\Leftrightarrow\; (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \;\Leftrightarrow\; r^2(\cos^2\theta - \sin^2\theta) = 1 \;\Leftrightarrow\; r^2\cos 2\theta = 1 \;\Rightarrow\; r^2 = \sec 2\theta$

3. (a)



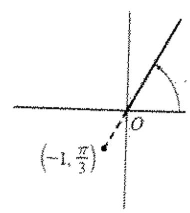
$x = 3\cos\frac{\pi}{2} = 3(0) = 0$ and
$y = 3\sin\frac{\pi}{2} = 3(1) = 3$ give us
the Cartesian coordinates $(0, 3)$.

(b)



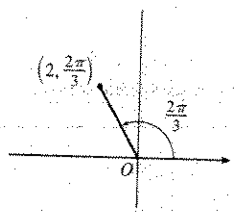
$x = 2\sqrt{2}\cos\frac{3\pi}{4}$
$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$ and
$y = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$
give us $(-2, 2)$.

(c)



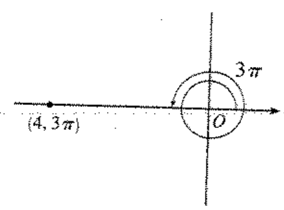
$x = -1\cos\frac{\pi}{3} = -$
$y = -1\sin\frac{\pi}{3} = -$
us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

4. (a)



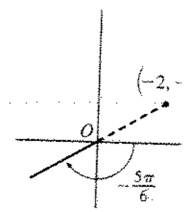
$x = 2\cos\frac{2\pi}{3} = -1$ and
$y = 2\sin\frac{2\pi}{3} = \sqrt{3}$ give
us $(-1, \sqrt{3})$.

(b)



$x = 4\cos 3\pi = -4$ and
$y = 4\sin 3\pi = 0$ give
us $(-4, 0)$.

(c)



$x = -2\cos(-\frac{5\pi}{6}) =$
and $y = -2\sin(-\frac{5\pi}{6}$
give us $(\sqrt{3}, 1)$.

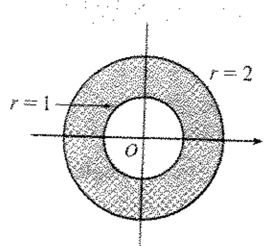
5. (a) $x = 1$ and $y = 1 \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the pol
coordinates are (i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

(b) $x = 2\sqrt{3}$ and $y = -2 \Rightarrow r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$ and
$\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $(2\sqrt{3}, -2)$ is in the fourth quadrant and $0 \leq \theta \leq 2\pi$, the pola
coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

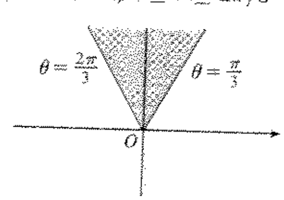
6. (a) $(x, y) = \left(-1, -\sqrt{3}\right)$, $r = \sqrt{1 + 3} = 2$, $\tan\theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$.
The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right.$
The polar coordinates are (i) $\left(\sqrt{13}, \theta\right)$ and (ii) $\left(-\sqrt{13}, \theta + \pi\right)$.

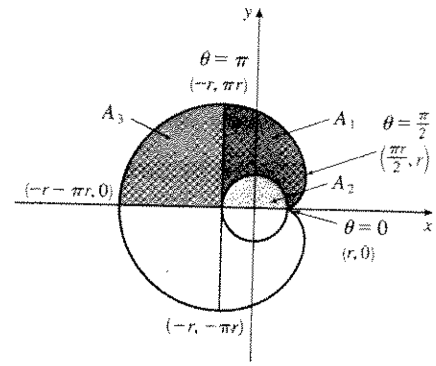
7. The curves $r = 1$ and $r = 2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \leq r \leq 2$ consists of both circles and the shaded region between them in the figure.



8. $r \geq 0, \quad \pi/3 \leq \theta \leq 2\pi/3$



x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.]

The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$.

Now $y\,dx = r(\sin\theta - \theta\cos\theta)r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate:

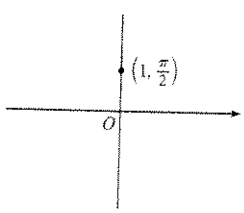
$(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$A_1 + A_2 = r^2\left[-\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta\right]_\pi^0 = r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right] = r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

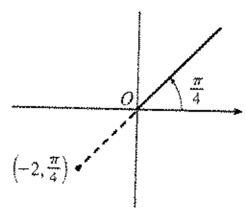
Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

9.3 Polar Coordinates

1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r < 0$ requirement.

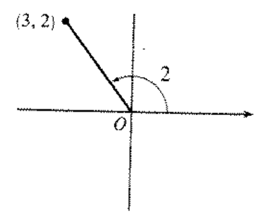


(b) $\left(-2, \frac{\pi}{4}\right)$



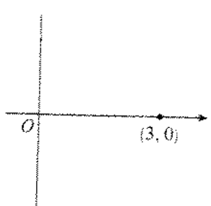
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3, 2)$



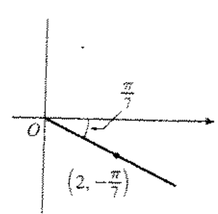
$(3, 2 + 2\pi), (-3, 2 + \pi)$

2. (a) $(3, 0)$



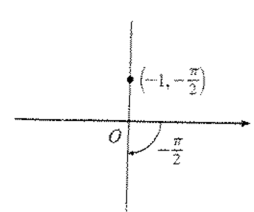
$(3, 2\pi), (-3, \pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$



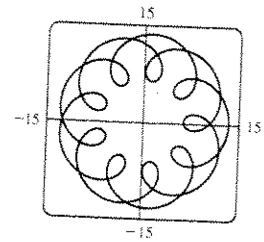
$\left(2, \frac{13\pi}{7}\right), \left(-2, \frac{6\pi}{7}\right)$

(c) $\left(-1, -\frac{\pi}{2}\right)$



$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

51. (a) Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \neq x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

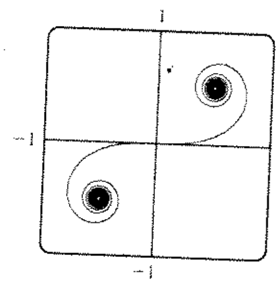


(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Formula 1 to find the arc length. Recent ver Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ as $88E(2\sqrt{2}\,i)$, where $E(x)$ is the elliptic integral $\int_0^1 \frac{\sqrt{1 - x^2t^2}}{\sqrt{1 - t^2}}\, dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that th length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\,\cos\left(\frac{11t}{2}\right) - 4\sin t\,\sin\left(\frac{11t}{2}\right) + 5}\, dt$.

52. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.

(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Formula 1, the length of the curve from the origin to the point with parameter value t is

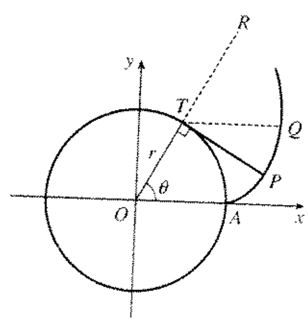
$$L = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_0^t \sqrt{\cos^2\left(\frac{\pi}{2}u^2\right) + \sin^2\left(\frac{\pi}{2}u^2\right)}\, du$$
$$= \int_0^t 1\, du = t \qquad [\text{or } -t \text{ if } t < 0]$$



We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

53. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates

$$x = r\cos\theta + r\theta\cos\left(\tfrac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta),$$
$$y = r\sin\theta - r\theta\sin\left(\tfrac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta).$$



54. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 53 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the

46. $x = 2a\cot\theta \Rightarrow dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta \Rightarrow dy/dt = 4a\sin\theta\cos\theta = 2a\sin 2\theta$.

So $L = \int_{\pi/4}^{\pi/2}\sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\,d\theta = 2a\int_{\pi/4}^{\pi/2}\sqrt{\csc^4\theta + \sin^2 2\theta}\,d\theta$. Using Simpson's Rule with $n = 4$,

$\Delta\theta = \frac{\pi/2-\pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$L \approx 2a\cdot S_4 = (2a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

47. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi$.

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \Rightarrow$

$$\begin{aligned}\text{Distance} &= \int_0^{3\pi}\sqrt{2}\,|\sin 2t|\,dt = 6\sqrt{2}\int_0^{\pi/2}\sin 2t\,dt \quad \text{[by symmetry]} = -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2}\\ &= -3\sqrt{2}(-1-1) = 6\sqrt{2}\end{aligned}$$

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2}\sin 2t\,dt = \sqrt{2}$, as above.

48. $x = \cos^2 t,\ y = \cos t,\ 0 \le t \le 4\pi$. $(dx/dt)^2 + (dy/dt)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t(4\cos^2 t + 1)$

$$\begin{aligned}\text{Distance} &= \int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi}\sin t\sqrt{4\cos^2 t + 1}\,dt\\ &= -4\int_1^{-1}\sqrt{4u^2+1}\,du \quad [u = \cos t,\ du = -\sin t\,dt] = 4\int_{-1}^{1}\sqrt{4u^2+1}\,du = 8\int_0^1\sqrt{4u^2+1}\,du\\ &= 8\int_0^{\tan^{-1}2}\sec\theta\cdot\tfrac{1}{2}\sec^2\theta\,d\theta \qquad [2u = \tan\theta,\ 2\,du = \sec^2\theta\,d\theta]\\ &= 4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta \overset{71}{=} \left[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}2} = 4\sqrt{5} + 2\ln(\sqrt{5}+2)\end{aligned}$$

Thus, $L = \int_0^{\pi}|\sin t|\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)$.

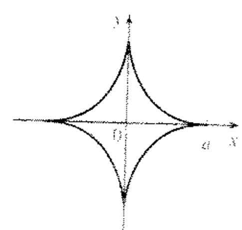
9. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi$.

$$\begin{aligned}(dx/d\theta)^2 + (dy/d\theta)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta\\ &= a^2 - (a^2 - b^2)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2}\sqrt{a^2(1 - e^2\sin^2\theta)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$.

$x = a\cos^3\theta,\ y = a\sin^3\theta$.

$$\begin{aligned}(dx/d\theta)^2 + (dy/d\theta)^2 &= (-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2\\ &= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta\\ &= 9a^2\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\cos^2\theta\end{aligned}$$



The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \le \theta \le \pi/2$. Thus,

$$\begin{aligned}L &= 4\int_0^{\pi/2}3a\sin\theta\cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \le \theta \le \pi/2]\\ &= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a\end{aligned}$$

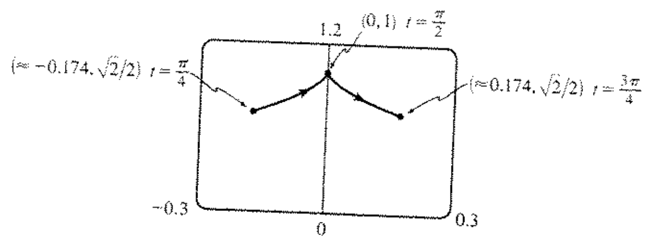
42. $x = \cos t + \ln\left(\tan \frac{1}{2}t\right)$, $y = \sin t$, $\pi/4 \le t \le 3\pi/4$.

$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\cos(t/2)} = -\sin t + \frac{1}{\sin t}$$

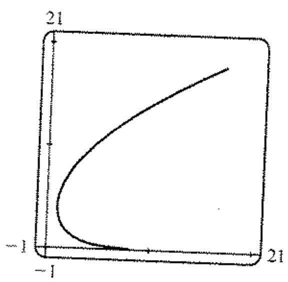
$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\cos(t/2)} = -\sin t + \frac{1}{\sin t}$$

and $\frac{dy}{dt} = \cos t$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t$. Thus,

$$L = \int_{\pi/4}^{3\pi/4} \left|\cot t\right| dt = 2\int_{\pi/4}^{\pi/2} \cot t \, dt = 2\left[\ln|\sin t|\right]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right)$$
$$= 2\left(0 + \ln\sqrt{2}\right) = 2\left(\tfrac{1}{2}\ln 2\right) = \ln 2$$



43.

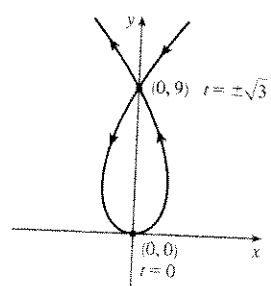


$x = e^t - t$, $y = 4e^{t/2}$, $-8 \le t \le 3$.

$$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t$$
$$= e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

$$L = \int_{-8}^{3} \sqrt{(e^t + 1)^2}\, dt = \int_{-8}^{3} (e^t + 1)\, dt = \left[e^t + t\right]_{-8}^{3}$$
$$= (e^3 + 3) - (e^{-8} - 8) = e^3 - e^{-8} + 11$$

44. $x = 3t - t^3$, $y = 3t^2$. $dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so $(dx/dt)^2 + (dy/dt)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$ and the length of the loop is given by



$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\, dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\, dt = 2[3t + t^3]_0^{\sqrt{3}}$$
$$= 2(3\sqrt{3} + 3\sqrt{3}) = 12\sqrt{3}$$

45. $x = t - e^t$, $y = t + e^t$, $-6 \le t \le 6$.

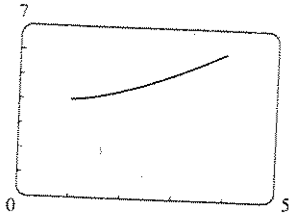
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = \left(1 - 2e^t + e^{2t}\right) + \left(1 + 2e^t + e^{2t}\right) = 2 + 2e^{2t}$, so $L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}$

Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-(-6)}{6} = 2$, we get $L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053$.

36. $x = \ln t,\ y = \sqrt{t+1},\ 1 \le t \le 5.$ $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t+1}}$, so $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{t^2} + \dfrac{1}{4(t+1)} = \dfrac{t^2+4t+4}{4t^2(t+1)}$.

Thus, $L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^5 \sqrt{\frac{t^2+4t+4}{4t^2(t+1)}}\,dt = \int_1^5 \sqrt{\frac{(t+2)^2}{(2t)^2(t+1)}}\,dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}}\,dt.$

37.



$x = 1 + 3t^2,\ \ y = 4 + 2t^3,\ \ 0 \le t \le 1.$

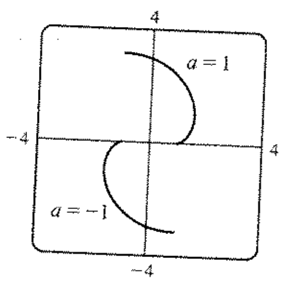
$dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$.

Thus, $L = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt$

$= \int_0^1 6t\sqrt{1+t^2}\,dt = 6\int_1^2 \sqrt{u}\,\left(\tfrac{1}{2}du\right) \quad [u = 1 + t^2,\ du = 2t\,dt]$

$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2(2\sqrt{2} - 1)$

38.



$x = a(\cos\theta + \theta\sin\theta),\ \ y = a(\sin\theta - \theta\cos\theta),\ \ 0 \le \theta \le \pi$

$\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = a^2[(-\sin\theta + \theta\cos\theta + \sin\theta)^2 + (\cos\theta + \theta\sin\theta - \cos\theta)^2]$

$= a^2\theta^2(\cos^2\theta + \sin^2\theta) = (a\theta)^2$

$L = \int_0^\pi a\theta\,d\theta = a\left[\tfrac{1}{2}\theta^2\right]_0^\pi = \tfrac{1}{2}\pi^2 a$

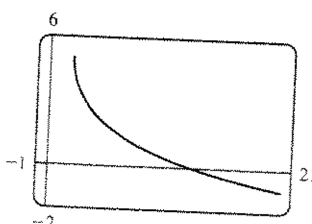
39. $x = \dfrac{t}{1+t},\ y = \ln(1+t),\ 0 \le t \le 2.$ $\dfrac{dx}{dt} = \dfrac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{1}{1+t}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2} = \dfrac{1}{(1+t)^4}\left[1 + (1+t)^2\right] = \dfrac{t^2+2t+2}{(1+t)^4}$. Thus,

$L = \displaystyle\int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du \ \ [u = t+1,\ du = dt] \ \overset{24}{=} \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\right]_1^3$

$= -\frac{\sqrt{10}}{3} + \ln\left(3 + \sqrt{10}\right) + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)$

40.



$x = e^t + e^{-t},\ y = 5 - 2t,\ 0 \le t \le 3.$

$dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$

and $L = \int_0^3 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}$.

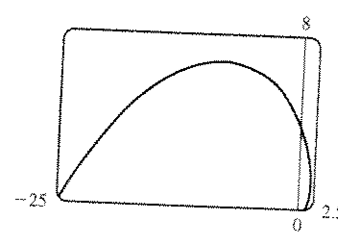
41. $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\sin t + \cos t)\right]^2$

$= \left(e^t\right)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)$

$+ \left(e^t\right)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t)$

$= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}$



Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\left[e^t\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

30. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int \sin^4\theta\cos^2\theta\,d\theta = \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$
$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

31. $A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta$
$= \left[r^2\theta - 2dr\sin\theta + \frac{1}{2}d^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$

32. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by x = t^2, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}}(t^3 - 3t)2t\,dt = 2\int_0^{-\sqrt{3}}(2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$
$$= 2\left[\tfrac{2}{5}(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac{2}{5}(-9\sqrt{3}) - 2(-3\sqrt{3})\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}}(t^3 - 3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt{3}}(t^6 - 6t^4 + 9t^2)t\,dt$$
$$= 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}} = 2\pi\left[\tfrac{1}{8}(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac{9}{4}(-3^{1/2})^4\right]$$
$$= 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coord of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 7.5.12 with $A = \frac{12}{5}\sqrt{}$ we get

$$\overline{x} = \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$
$$= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}(-3^{1/2})^7 - \tfrac{3}{5}(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

33. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1 - 2t$ and $dy/dt = 2t^{1/2}$, so
$(dx/dt)^2 + (dy/dt)^2 = (1 - 2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$. Thus,
$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^2 \sqrt{1 + 4t^2}\,dt$.

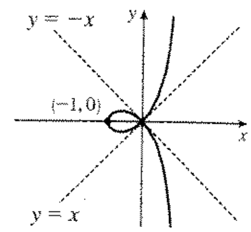
34. $x = 1 + e^t$, $y = t^2$, $-3 \le t \le 3$. $dx/dt = e^t$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2$. Thus,
$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_{-3}^3 \sqrt{e^{2t} + 4t^2}\,dt$.

35. $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$. $dx/dt = 1 - \sin t$ and $dy/dt = 1 - \cos t$, so

$$(dx/dt)^2 + (dy/dt)^2 = (1 - \sin t)^2 + (1 - \cos t)^2 = (1 - 2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t)$$
$$= 3 - 2\sin t - 2\cos t$$

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi}\sqrt{3 - 2\sin t - 2\cos t}\,dt$.

Taking $t_1 = \frac{\pi}{4}$ and $t_2 = -\frac{\pi}{4}$ gives $(x, y) = (0, 0)$ for both values of t. $dx/dt = 2\sin 2t$, and $dy/dt = 2\sin 2t \tan t - \cos 2t \sec^2 t$. When $t = \frac{\pi}{4}$, $dx/dt = 2$ and $dy/dt = 2$, so $dy/dx = 1$. When $t = -\frac{\pi}{4}$, $dx/dt = -2$ and $dy/dt = 2$, so $dy/dx = -1$. Thus, the equations of the two tangents at $(0, 0)$ are $y = x$ and $y = -x$.



23. (a) $x = r\theta - d\sin\theta,\ y = r - d\cos\theta.$ $\dfrac{dx}{d\theta} = r - d\cos\theta,\ \dfrac{dy}{d\theta} = d\sin\theta.$ So $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

24. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow$ $dy/dx = 0$ $\Leftrightarrow$ $\tan\theta = 0$ $\Leftrightarrow$ $\theta = n\pi$ $\Leftrightarrow$ $(x, y) = (\pm a, 0)$.

The tangent is vertical $\Leftrightarrow$ $\cos\theta = 0$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{2}$ $\Leftrightarrow$ $(x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1$ $\Leftrightarrow$ $\tan\theta = \pm 1$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{4}$ $\Leftrightarrow$ $(x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

(All sign choices are valid.)

25. The line with parametric equations $x = -7t,\ y = 12t - 5$ is $y = 12\left(-\frac{1}{7}x\right) - 5$, which has slope $-\frac{12}{7}$.

The curve $x = t^3 + 4t,\ y = 6t^2$ has slope $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t}{3t^2 + 4}$. This equals $-\frac{12}{7}$ $\Leftrightarrow$ $3t^2 + 4 = -7t$ $\Leftrightarrow$ $(3t + 4)(t + 1) = 0$ $\Leftrightarrow$ $t = -1$ or $t = -\frac{4}{3}$ $\Leftrightarrow$ $(x, y) = (-5, 6)$ or $\left(-\frac{208}{27}, \frac{32}{3}\right)$.

26. $x = 3t^2 + 1,\ y = 2t^3 + 1.$ $\dfrac{dx}{dt} = 6t,\ \dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ (even where $t = 0$).

So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t\left[x - (3t^2 + 1)\right]$. If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t\left[4 - (3t^2 + 1)\right]$ $\Leftrightarrow$ $2t^3 - 2 = 3t^3 - 3t$ $\Leftrightarrow$ $t^3 - 3t + 2 = 0$ $\Leftrightarrow$ $(t - 1)^2(t + 2) = 0$ $\Leftrightarrow$ $t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

27. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

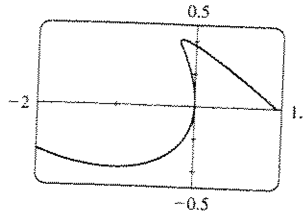
28. $t + 1/t = 2.5$ $\Leftrightarrow$ $t = \frac{1}{2}$ or 2, and for $\frac{1}{2} < t < 2$, we have $t + 1/t < 2.5$. $x = -\frac{3}{2}$ when $t = \frac{1}{2}$ and $x = \frac{3}{2}$ when $t = 2$.

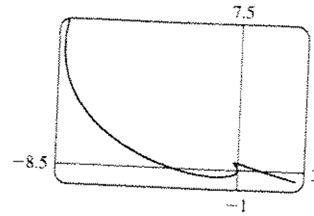
$$A = \int_{-3/2}^{3/2}(2.5 - y)\,dx = \int_{1/2}^{2}\left(\tfrac{5}{2} - t - 1/t\right)(1 + 1/t^2)\,dt \qquad [x = t - 1/t,\ dx = (1 + 1/t^2)\,dt]$$
$$= \int_{1/2}^{2}\left(-t + \tfrac{5}{2} - 2t^{-1} + \tfrac{5}{2}t^{-2} - t^{-3}\right)dt = \left[\frac{-t^2}{2} + \frac{5t}{2} - 2\ln|t| - \frac{5}{2t} + \frac{1}{2t^2}\right]_{1/2}^{2}$$
$$= \left(-2 + 5 - 2\ln 2 - \tfrac{5}{4} + \tfrac{1}{8}\right) - \left(-\tfrac{1}{8} + \tfrac{5}{4} + 2\ln 2 - 5 + 2\right) = \tfrac{15}{4} - 4\ln 2$$

29. $A = \int_0^1 (y - 1)\,dx = \int_{\pi/2}^0 (e^t - 1)(-\sin t)\,dt = \int_0^{\pi/2}(e^t\sin t - \sin t)\,dt \overset{98}{=} \left[\frac{1}{2}e^t(\sin t - \cos t) + \cos t\right]_0^{\pi/2}$

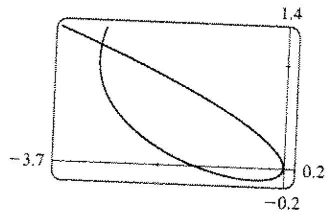
$= \frac{1}{2}(e^{\pi/2} - 1)$

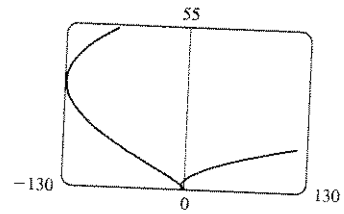
19. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$. We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \Leftrightarrow t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t(2t^2 - 3t - 2) = 0 \Leftrightarrow 2t(2t + 1)(t - 2) = 0 \Leftrightarrow t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.



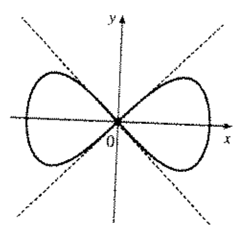


20. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$. We calculate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t - 1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \Leftrightarrow t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \Leftrightarrow 4t(t^2 + 3t - 4) = 0 \Leftrightarrow 4t(t + 4)(t - 1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.



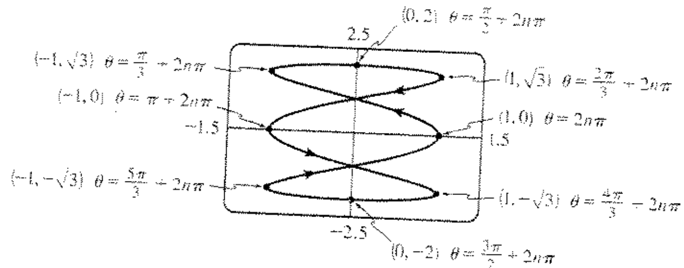


21. $x = \cos t$, $y = \sin t \cos t$. $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = 1$. When $t = \frac{3\pi}{2}$, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1$. So $\frac{dy}{dx} = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.

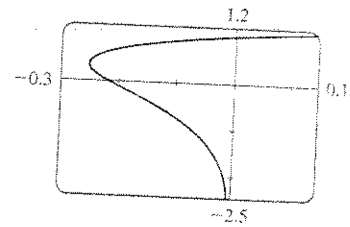


22. $x = 1 - 2\cos^2 t = -\cos 2t$, $y = (\tan t)(1 - 2\cos^2 t) = -(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x, y). Call these values t_1 and t_2. Then $\cos^2 t_1 = \cos^2 t_2$ (from the equation for x) and either $\tan t_1 = \tan t_2$ or $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1 = \cos^2 t_2$ and $\tan t_1 = \tan t_2$ by choosing t_1 arbitrarily and taking $t_2 = t_1 + \pi$, so evidently the whole curve is retraced every time t traverses an interval of length π. Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $t_2 = -t_1$, then $\cos^2 t_2 = \cos^2 t_1$, but $\tan t_2 = -\tan t_1$. This suggests that we try to satisfy the condition $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$.

16. $x = \cos 3\theta$, $y = 2\sin\theta$. $dy/d\theta = 2\cos\theta$, so $dy/d\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow (x, y) = (0, \pm 2)$. Also, $dx/d\theta = -3\sin 3\theta$, so $dx/d\theta = 0 \Leftrightarrow 3\theta = n\pi \Leftrightarrow \theta = \frac{\pi}{3}n \Leftrightarrow (x, y) = (\pm 1, 0)$ or $(\pm 1, \pm\sqrt{3})$. The curve has horizontal tangents at $(0, \pm 2)$, and vertical tangents at $(\pm 1, 0)$, $(\pm 1, -\sqrt{3})$ and $(\pm 1, \sqrt{3})$.

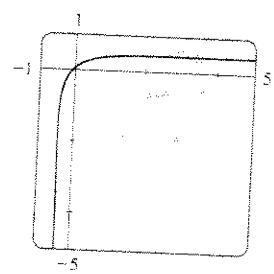


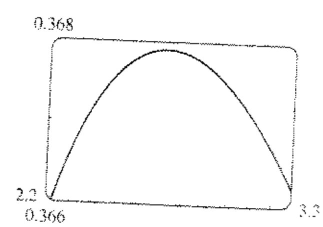
17. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 4t^3 - 2t \Leftrightarrow 2t(2t^2 - 1) = 0 \Leftrightarrow$ $2t(\sqrt{2}t + 1)(\sqrt{2}t - 1) = 0 \Leftrightarrow t = 0$ or $\pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be



$$\left(x\left(\frac{1}{\sqrt{2}}\right), y\left(\frac{1}{\sqrt{2}}\right)\right) = \left(\left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2, \frac{1}{\sqrt{2}} + \ln\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{4}, \frac{1}{\sqrt{2}} - \frac{1}{2}\ln 2\right)$$

. The curve is symmetric about the line $y = -x$ since replacing t with $-t$ has the effect of replacing (x, y) with $(-y, -x)$, so if we can find the highest point (x_h, y_h), then the leftmost point is $(x_l, y_l) = (-y_h, -x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x = te^t$, $y = te^{-t}$ is about $(2.7, 0.37)$.





To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt = 0 \Leftrightarrow t(-e^{-t}) + e^{-t} = 0 \Leftrightarrow (1 - t)e^{-t} = 0 \Leftrightarrow t = 1$. This corresponds to the point $(x(1), y(1)) = (e, 1/e)$. To find the leftmost point, we find the value of t for which $0 = dx/dt = te^t + e^t \Leftrightarrow$ $(1 + t)e^t = 0 \Leftrightarrow t = -1$. This corresponds to the point $(x(-1), y(-1)) = (-1/e, -e)$.

As $t \to -\infty$, $x(t) = te^t \to 0^-$ by l'Hospital's Rule and $y(t) = te^{-t} \to -\infty$, so the y-axis is an asymptote. As $t \to \infty$, $x(t) \to \infty$ and $y(t) \to 0^+$, so the x-axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t-interval.

9.2 Calculus with Parametric Curves

1. $x = t - t^3$, $y = 2 - 5t \Rightarrow \dfrac{dy}{dt} = -5$, $\dfrac{dx}{dt} = 1 - 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-5}{1-3t^2}$ or $\dfrac{5}{3t^2-1}$.

2. $x = te^t$, $y = t + e^t \Rightarrow \dfrac{dy}{dt} = 1 + e^t$, $\dfrac{dx}{dt} = te^t + e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+e^t}{te^t + e^t}$.

3. $x = t^4 + 1$, $y = t^3 + t$; $t = -1$. $\dfrac{dy}{dt} = 3t^2 + 1$, $\dfrac{dx}{dt} = 4t^3$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2+1}{4t^3}$. When $t = -1$, $(x, y) = (2, -2)$ and $dy/dx = \frac{4}{-4} = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - (-2) = (-1)(x - 2)$, or $y = -x$.

4. $x = 2t^2 + 1$, $y = \frac{1}{3}t^3 - t$; $t = 3$. $\dfrac{dy}{dt} = t^2 - 1$, $\dfrac{dx}{dt} = 4t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^2-1}{4t}$. When $t = 3$, $(x, y) = (19, 6)$ and $dy/dx = \frac{8}{12} = \frac{2}{3}$, so an equation of the tangent line is $y - 6 = \frac{2}{3}(x - 19)$, or $y = \frac{2}{3}x - \frac{20}{3}$.

5. $x = e^{\sqrt{t}}$, $y = t - \ln t^2$; $t = 1$. $\dfrac{dy}{dt} = 1 - \dfrac{2t}{t^2} = 1 - \dfrac{2}{t}$, $\dfrac{dx}{dt} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2/t}{e^{\sqrt{t}}/(2\sqrt{t})} \cdot \dfrac{2t}{2t} = \dfrac{2t-4}{\sqrt{t}\,e^{\sqrt{t}}}$.
When $t = 1$, $(x, y) = (e, 1)$ and $\dfrac{dy}{dx} = -\dfrac{2}{e}$, so an equation of the tangent line is $y - 1 = -\frac{2}{e}(x - e)$, or $y = -\frac{2}{e}x + 3$.

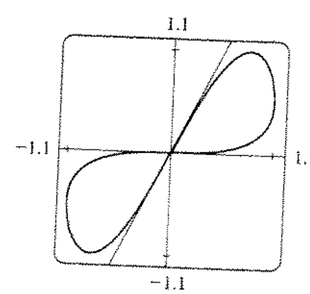
6. $x = \cos\theta + \sin 2\theta$, $y = \sin\theta + \cos 2\theta$; $\theta = 0$. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - 2\sin 2\theta}{-\sin\theta + 2\cos 2\theta}$. When $\theta = 0$, $(x, y) = (1, 1)$ and $dy/dx = \frac{1}{2}$, so an equation of the tangent to the curve is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

7. (a) $x = e^t$, $y = (t-1)^2$; $(1, 1)$. $\dfrac{dy}{dt} = 2(t-1)$, $\dfrac{dx}{dt} = e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2(t-1)}{e^t}$.
At $(1, 1)$, $t = 0$ and $\dfrac{dy}{dx} = -2$, so an equation of the tangent is $y - 1 = -2(x - 1)$, or $y = -2x + 3$.

(b) $x = e^t \Rightarrow t = \ln x$, so $y = (t-1)^2 = (\ln x - 1)^2$ and $\dfrac{dy}{dx} = 2(\ln x - 1)\left(\dfrac{1}{x}\right)$. When $x = 1$, $\dfrac{dy}{dx} = 2(-1)(1) = -2$, so an equation of the tangent is $y = -2x + 3$, as in part (a).

8. $x = \sin t$, $y = \sin(t + \sin t)$; $(0, 0)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t + \sin t)(1 + \cos t)}{\cos t} = (\sec t + 1)\cos(t + \sin t)$$



Note that there are two tangents at the point $(0, 0)$, since both $t = 0$ and $t = \pi$ correspond to the origin. The tangent corresponding to $t = 0$ has slope $(\sec 0 + 1)\cos(0 + \sin 0) = 2\cos 0 = 2$, and its equation is $y = 2x$. The tangent corresponding to $t = \pi$ has slope $(\sec\pi + 1)\cos(\pi + \sin\pi) = 0$, so it is the x-axis.

9. $x = 4 + t^2$, $y = t^2 + t^3 \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 3t^2}{2t} = 1 + \dfrac{3}{2}t \Rightarrow$
$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d(dy/dx)/dt}{dx/dt} = \dfrac{(d/dt)(1 + \frac{3}{2}t)}{2t} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > 0$.