8 ☐ SERIES

8.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
   (b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
   (c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$
   (b) A divergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. The first six terms of $a_n = \dfrac{n}{2n+1}$ are $\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}, \dfrac{5}{11}, \dfrac{6}{13}$. It appears that the sequence is approaching $\dfrac{1}{2}$.
$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2}$$

4. $\{\cos(n\pi/3)\}_{n=1}^{9} = \left\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1\right\}$. The sequence does not appear to have a limit. The values will cycle through the first six numbers in the sequence—never approaching a particular number.

5. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

6. $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

7. $\{2, 7, 12, 17, \ldots\}$. Each term is larger than the preceding one by 5, so $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

8. $\{5, 1, 5, 1, 5, 1, \ldots\}$. The average of 5 and 1 is 3, so we can think of the sequence as alternately adding 2 and -2 to 3. Thus, $a_n = 3 + (-1)^{n+1} \cdot 2$.

9. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

10. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

11. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3} \lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

12. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

13. $a_n = \dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

14. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0 + 1 = 1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

15. $a_n = \dfrac{(-1)^{n-1}n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 6. Converges

16. $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $|a_n| = \dfrac{n^3}{n^3+2n^2+1} = \dfrac{1}{1+\frac{2}{n}+\frac{1}{n^3}} \to 1$ as $n \to \infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1, a_3, a_5, \ldots$ converges to -1 and the sequence $a_2, a_4, a_6, \ldots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

17. $a_n = \dfrac{e^n + e^{-n}}{e^{2n}-1} \cdot \dfrac{e^{-n}}{e^{-n}} = \dfrac{1+e^{-2n}}{e^n - e^{-n}} \to \dfrac{1+0}{e^n - 0} \to 0$ as $n \to \infty$. Converges

18. $a_n = \cos(2/n)$. As $n \to \infty$, $2/n \to 0$, so $\cos(2/n) \to \cos 0 = 1$. Converges

19. $a_n = n^2 e^{-n} = \dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty} a_n = 0$. Converges

20. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Converges

21. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

22. $a_n = n\cos n\pi = n(-1)^n$. Since $|a_n| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

23. $y = \left(1+\dfrac{2}{x}\right)^x \Rightarrow \ln y = x\ln\left(1+\dfrac{2}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\left(\dfrac{1}{1+2/x}\right)\left(-\dfrac{2}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2 \Rightarrow$$

$\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln y} = e^2$, so by Theorem 2, $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^n = e^2$. Convergent

24. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$. $|a_n| \le \dfrac{1}{1+\sqrt{n}}$ and $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, so $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

25. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

26. $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{2(\ln x)(1/x)}{1} = 2\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{H}{=} 2\lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so by Theorem 3, $\lim\limits_{n\to\infty} \dfrac{(\ln n)^2}{n} = 0$. Convergent

27. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\dfrac{2n^2+1}{n^2+1}\right) = \ln\left(\dfrac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$ as $n \to \infty$. Convergent

28. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \le \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{n}$ [for $n > 2$] $= \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n\}$ converges to 0.

29. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

39. We show by induction that $\{a_n\}$ is increasing and bounded above by 3. Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true.

Then $a_{n+1} > a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n}$. Now $a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1} \Leftrightarrow P_{n+1}$.

This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy

$L = 3 - 1/L \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3\pm\sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3+\sqrt{5}}{2}$.

40. We use induction. Let P_n be the statement that $0 < a_{n+1} \le a_n \le 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \le a_n \Rightarrow -a_{n+1} \ge -a_n \Rightarrow 3 - a_{n+1} \ge 3 - a_n \Rightarrow$

$a_{n+2} = \frac{1}{3 - a_{n+1}} \le \frac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \le 2$ by the induction hypothesis, so P_{n+1} is true. To find the limit, we use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \Rightarrow L = \frac{1}{3-L} \Rightarrow$

$L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3\pm\sqrt{5}}{2}$. But $L \le 2$, so we must have $L = \frac{3-\sqrt{5}}{2}$.

41. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \frac{f_{n+1}}{f_n} \Rightarrow a_{n-1} = \frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + \frac{1}{f_{n-1}/f_{n-2}} = 1 + \frac{1}{a_{n-2}}$. If $L = \lim_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{n-1}$ and $L = \lim_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1+\sqrt{5}}{2}$ (since L must be positive).

42. (a) If f is continuous, then $f(L) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = L$ by Exercise 38(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

43. $(0.8)^n < 0.000001 \Rightarrow \ln(0.8)^n < \ln(0.000001) \Rightarrow n\ln(0.8) < \ln(0.000001) \Rightarrow n > \frac{\ln(0.000001)}{\ln(0.8)} \Rightarrow$ $n > 61.9$, so n must be at least 62 to satisfy the given inequality.

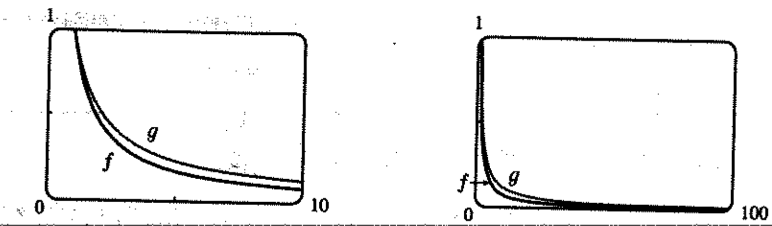
44. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$ then $n > \ln(\varepsilon)/\ln|r| \Rightarrow$ $n\ln|r| < \ln\varepsilon$ [since $|r| < 1 \Rightarrow \ln|r| < 0$] $\Rightarrow \ln(|r|^n) < \ln\varepsilon \Rightarrow |r|^n < \varepsilon \Rightarrow |r^n - 0| < \varepsilon$, and so by Definition 2, $\lim_{n\to\infty} r^n = 0$.

45. If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \le a_n \le |a_n|$, we have that $\lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

46. (a) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim_{n\to\infty} a_n = L$.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is
$A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is
$A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for
$1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is
$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)



(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\frac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)dx = \int_1^t x^{-0.9}dx = \left[\frac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\, du = 3\,dx] \quad = -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$$

so $I = \lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}$. Convergent

6. $\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to-\infty}\left[\frac{1}{2}\ln 5 - \frac{1}{2}\ln|2t-5|\right] = -\infty$. Divergent

7. $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$ $\quad [u = 2-w,\, du = -dw]$
$= \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty$. Divergent

8. $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right) = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$.
Convergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}(-2e^{-t/2}+2e^{-2}) = 0+2e^{-2} = 2e^{-2}$.   Convergent

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty}\left[-\frac{1}{2}e^{2}+\frac{1}{2}e^{-2x}\right] = \infty$.   Divergent

11. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim_{t\to\infty}(-\cos t+1)$.

This limit does not exist, so the integral is divergent.   Divergent

12. $I = \int_{-\infty}^\infty (2-v^4)\,dv = I_1 + I_2 = \int_{-\infty}^0 (2-v^4)\,dv + \int_0^\infty (2-v^4)\,dv$, but

$I_1 = \lim_{t\to-\infty}\left[2v-\frac{1}{5}v^5\right]_t^0 = \lim_{t\to-\infty}\left(-2t+\frac{1}{5}t^5\right) = -\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2.   Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$.

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1-e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}$. Therefore,

$\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0$.   Convergent

14. $\int_{-\infty}^\infty x^2e^{-x^3}\,dx = \int_{-\infty}^0 x^2e^{-x^3}\,dx + \int_0^\infty x^2e^{-x^3}\,dx$, and

$\int_{-\infty}^0 x^2e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0 = -\frac{1}{3}+\frac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right) = \infty$.   Divergent

15. $\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s}-\frac{1}{25}e^{-5s}\right]_0^t$ [by integration by parts with $u=s$]

$= \lim_{t\to\infty}\left(-\frac{1}{5}te^{-5t}-\frac{1}{25}e^{-5t}+\frac{1}{25}\right) = 0-0+\frac{1}{25}$ [by l'Hospital's Rule]

$= \frac{1}{25}$ Convergent

16. $I = \int_{-\infty}^\infty \cos\pi t\,dt = I_1 + I_2 = \int_{-\infty}^0 \cos\pi t\,dt + \int_0^\infty \cos\pi t\,dt$, but $I_1 = \lim_{s\to-\infty}\left[\frac{1}{\pi}\sin\pi t\right]_s^0 = \lim_{s\to-\infty}\left(-\frac{1}{\pi}\sin\pi t\right)$

and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2.   Divergent

17. $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ [by substitution with $u=\ln x$, $du = dx/x$] $= \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$.   Divergent

18. $\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3}-9e^{r/3}\right]_t^6$ [by integration by parts with $u=r$]

$= \lim_{t\to-\infty}(18e^2-9e^2-3te^{t/3}+9e^{t/3}) = 9e^2-0+0$ [by l'Hospital's Rule]

$= 9e^2$ Convergent

19. Integrate by parts with $u = \ln x$, $dv = dx/x^2 \;\Rightarrow\; du = dx/x$, $v = -1/x$.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1\right) = -0 - 0 + 0 + 1 = 1$$

since $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1/t}{1} = 0.$ Convergent

20. Integrate by parts with $u = \ln x$, $dv = dx/x^3 \;\Rightarrow\; du = dx/x$, $v = -1/(2x^2)$.

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left(\left[-\frac{1}{2x^2}\ln x\right]_1^t + \frac{1}{2}\int_1^t \frac{1}{x^3}\,dx\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4}\right) = \frac{1}{4}$$

since $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t^2} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1/t}{2t} = \lim_{t\to\infty}\frac{1}{2t^2} = 0.$ Convergent

21. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\int_0^\infty \frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\displaystyle\int \frac{x^2\,dx}{9+x^6}\;\begin{bmatrix} u = x^3 \\ du = 3x^2\,dx\end{bmatrix} = \int \frac{\frac{1}{3}\,du}{9+u^2}\;\begin{bmatrix} u = 3v \\ du = 3\,dv\end{bmatrix} = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$$= \frac{1}{9}\tan^{-1}v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C,$$

so $\displaystyle 2\int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t = 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}.$

Convergent

22. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{(e^x)^2 + (\sqrt{3})^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{\sqrt{3}}\arctan\frac{e^x}{\sqrt{3}}\right]_0^t = \frac{1}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\frac{e^t}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{9}. \quad \text{Convergent}$$

23. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+}\int_t^1 3x^{-5}\,dx = \lim_{t\to 0^+}\left[-\frac{3}{4x^4}\right]_t^1 = -\frac{3}{4}\lim_{t\to 0^+}\left(1 - \frac{1}{t^4}\right) = \infty.$ Divergent

24. $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\int_2^t (3-x)^{-1/2}\,dx = \lim_{t\to 3^-}\left[-2(3-x)^{1/2}\right]_2^t = -2\lim_{t\to 3^-}\left(\sqrt{3-t} - \sqrt{1}\right) = -2(0-1) = 2.$

Convergent

25. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14} (x+2)^{-1/4}\,dx = \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[16^{3/4} - (t+2)^{3/4}\right]$

$$= \tfrac{4}{3}(8-0) = \tfrac{32}{3} \quad \text{Convergent}$$

26. $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx = \lim_{t\to 6^+}\int_t^8 4(x-6)^{-3}\,dx = \lim_{t\to 6^+}\left[-2(x-6)^{-2}\right]_t^8 = -2\lim_{t\to 6^+}\left[\frac{1}{2^2} - \frac{1}{(t-6)^2}\right] = \infty.$ Divergent

43. For $x \geq 1$, $x + e^{2x} > e^{2x} > 0 \quad \Rightarrow \quad \dfrac{1}{x + e^{2x}} \leq \dfrac{1}{e^{2x}} = e^{-2x}$ on $[1, \infty)$.

$\int_1^\infty e^{-2x}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_1^t = \lim\limits_{t\to\infty} \left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2}\right] = \frac{1}{2}e^{-2}$. Therefore, $\int_1^\infty e^{-2x}\,dx$ is convergent, and by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x + e^{2x}}$ is also convergent.

44. For $x \geq 1$, $0 < \dfrac{x}{\sqrt{1+x^6}} < \dfrac{x}{\sqrt{x^6}} = \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (2) with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ is convergent by the Comparison Theorem.

45. $\dfrac{1}{x \sin x} \geq \dfrac{1}{x}$ on $\left(0, \frac{\pi}{2}\right]$ since $0 \leq \sin x \leq 1$. $\displaystyle\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+} \int_t^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+} \left[\ln x\right]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\displaystyle\int_0^{\pi/2} \frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\displaystyle\int_0^{\pi/2} \frac{dx}{x \sin x}$ is also divergent.

46. For $0 \leq x \leq 1$, $e^{-x} \leq 1 \quad \Rightarrow \quad \dfrac{e^{-x}}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$.

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2$ is convergent.

Therefore, $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent by the Comparison Theorem.

47. $$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}\,(1+x)}.$$

Now $$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} \qquad [u = \sqrt{x},\ x = u^2,\ dx = 2u\,du]$$

$$= 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C,$$

so $$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+} \left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty} \left[2\tan^{-1}\sqrt{x}\right]_1^t$$

$$= \lim_{t\to 0^+} \left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty} \left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.$$

48. Let $u = \ln x$. Then $du = dx/x \quad \Rightarrow \quad \displaystyle\int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p > 1$ and diverges otherwise.

49. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \left[\ln x\right]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ $\left[\begin{array}{l}\text{note that the integral is}\\ \text{not improper if } p < 0\end{array}\right]$ $\displaystyle = \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+} \left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.