Homework 7 Solutions

8.2

4) $r = \frac{a_{n+1}}{a_n}$ $\quad \frac{.4}{1} = .4 \quad \frac{.16}{.4} = .4 \quad \frac{.064}{.16} = .4$ etc.

So $1 + .4 + .16 + .064 + \cdots = \sum_{n=1}^{\infty} 1 \cdot (.4)^{n-1}$

$|.4| < 1$, so by the formula on pg. 424

$\sum_{n=1}^{\infty} 1 \cdot (.4)^{n-1} = \frac{1}{1-.4} = \frac{1}{.6} = \boxed{\frac{5}{3}}$

7) $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3 \cdot 3^n} = \frac{1}{3} \sum_{n=0}^{\infty} (\pi/3)^n$

$\pi = 3.141\ldots > 3$. So $\pi/3 > 1$. Therefore $\frac{1}{3} \sum_{n=0}^{\infty} (\pi/3)^n$ $\left(= \sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}\right)$ $\boxed{\text{diverges}}$ to ∞.

9) $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} \cdot$ the harmonic series of Example 7.

The harmonic series diverges to infinity so $\sum_{n=1}^{\infty} \frac{1}{2n}$ $\boxed{\text{diverges}}$ to infinity.

10) $\lim_{n \to \infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$ (This is a rule from last semester about limits of rational functions. If the highest powers of n or x in the numerator and denominator are the same, then $\lim_{n \to \infty} \frac{a n^k + \cdots}{b n^k + \cdots} = \frac{a}{b}$)

By the Test for Divergence (page 427) $\sum \frac{n+1}{2n-3}$ $\boxed{\text{diverges}}$

13) $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

$= \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$

(Geometric Series)

15) $\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 1 \neq 0$. By the test for divergence (page 427), $\sum_{n=1}^{\infty} \sqrt[n]{2}$ $\boxed{\text{diverges}}$.

19) As in Example 6, we use partial fractions.

$$\frac{2}{(n^2-1)} = \frac{2}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1} \iff 2 = A(n-1) + B(n+1)$$

$$\iff \begin{array}{l} A+B=0 \\ -A+B=2 \end{array} \iff \begin{array}{l} B=1 \\ A=-1 \end{array} \iff \frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}$$

Let's examine some partial sums.

$S_2 = 1 - \frac{1}{3}$

$S_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_4 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$

$S_5 = 1 - \cancel{1/3} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$

$S_6 = 1 - \cancel{1/3} + 1/2 - \cancel{1/4} + \cancel{1/3} - \cancel{1/5} + \cancel{1/4} - 1/6 + \cancel{1/5} - 1/7$

So we see that $1, \frac{1}{2}$, the last term, and the third to last terms always remain. In short, $S_K = 1 + \frac{1}{2} - \frac{1}{K} - \frac{1}{K+1}$

Recall that by definition, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{K\to\infty} S_K = \lim_{K\to\infty} \frac{3}{2} - \frac{1}{K} - \frac{1}{K+1}$

$= \frac{3}{2} - \lim_{K\to\infty} \frac{1}{K} - \lim_{K\to\infty} \frac{1}{K+1}$ (because we know $\lim_{K\to\infty} \frac{1}{K}$ and $\lim_{K\to\infty} \frac{1}{K+1}$ both exist.)

$\boxed{= \frac{3}{2}}$

27) Let $a_n = x^n/3^n$. $a_{n+1} = x^{n+1}/3^{n+1}$ We can see that $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is geometric with common ratio $a_{n+1}/a_n = x/3 = r$

We know (page 424 in the book) that the series converges if and only if $|r| < 1 \iff |\frac{x}{3}| = \frac{|x|}{3} < 1 \iff \boxed{|x| < 3}$

$\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \frac{x}{3} + \frac{x^2}{9} + \cdots$ So $a = x/3$, $r = x/3$, and $\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \frac{x/3}{1-x/3}$

$\boxed{= \frac{x}{3-x}}$ in the cases in which the sum converges.

40) We use Theorem 6 and the Test for Divergence (pg 426-7)

By Theorem 6, $\lim_{n\to\infty} a_n = 0$. Then $\lim_{n\to\infty} \frac{1}{a_n}$ goes off to ∞

By the Test for Divergence, $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

46) By definition, $f_n = f_{n-1} + f_{n-2}$. Therefore $f_{n+1} = f_n + f_{n-1}$ and $f_n = f_{n+1} - f_{n-1}$ for $n \geq 3$

a) $f_n = f_{n+1} - f_{n-1}$. Divide both sides by $f_{n-1} f_n f_{n+1}$.

$$\frac{\cancel{f_n}}{f_{n-1}\cancel{f_n}f_{n+1}} = \frac{\cancel{f_{n+1}}}{f_{n-1}f_n\cancel{f_{n+1}}} - \frac{\cancel{f_{n-1}}}{\cancel{f_{n-1}}f_nf_{n+1}}$$

$$\iff \frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}} \quad \checkmark$$

To be complete, we should check this result for $n = 2$.

$\frac{1}{f_1 f_3} = \frac{1}{1\cdot 2} = \frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} = 1 - \frac{1}{2} = \frac{1}{2}$ ✓

b) $\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = \sum \frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}}$ (using part a))

Good thing we checked this case!

Examining partial sums:

$s_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$s_3 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{6}$

$s_4 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \frac{1}{30}$

So in general, $s_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cdots + \cancel{\frac{1}{k}} - \cancel{\frac{1}{k}} + \cdots - \cancel{\frac{1}{f_{n-1} f_{n}}} + \cancel{\frac{1}{f_{n-1} f_n}} - \frac{1}{f_n f_{n+1}}$
$= 1 - \frac{1}{f_n f_{n+1}}$

We could prove this by induction, but we are lazy.

$\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} 1 - \frac{1}{f_k f_{k+1}} = \boxed{1}$ because $f_k f_{k+1} \to \infty$

c) $\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_n} - \frac{f_n}{f_{n+1} f_n} = \sum_{n=2}^{\infty} \frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}$

$s_k = \frac{1}{f_1} - \cancel{\frac{1}{f_3}} + \frac{1}{f_2} - \cancel{\frac{1}{f_4}} + \cancel{\frac{1}{f_3}} - \frac{1}{f_5} + \cancel{\frac{1}{f_4}} - \frac{1}{f_6} + \cdots + \cancel{\frac{1}{f_{k-3}}} - \cancel{\frac{1}{f_{k-1}}}$
$+ \cancel{\frac{1}{f_{k-2}}} - \frac{1}{f_k} + \cancel{\frac{1}{f_{k-1}}} - \frac{1}{f_{k+1}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_k} - \frac{1}{f_{k+1}}$

Can you see how this is similar to 1a) in particular?

$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{f_1} + \frac{1}{f_2} - \underbrace{\frac{1}{f_k}}_{\to 0} - \underbrace{\frac{1}{f_{k+1}}}_{\to 0} = \frac{1}{1} + \frac{1}{1}$

$= \boxed{2}$