6. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^\infty x^{-1/4}\,dx = \lim\limits_{t\to\infty}\int_1^t x^{-1/4}\,dx = \lim\limits_{t\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^t = \lim\limits_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^\infty 1/\sqrt[4]{n}$ diverges.

7. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty \frac{1}{x^4}\,dx = \lim_{t\to\infty}\int_1^t x^{-4}\,dx = \lim_{t\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4}$ is also convergent by the Integral Test.

8. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_1^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges.

9. $\displaystyle\frac{1}{n^2+n+1} < \frac{1}{n^2}$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n+1}$ converges by comparison with $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

10. $\displaystyle\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\displaystyle\sum_{n=2}^\infty \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p-series $\displaystyle\sum_{n=2}^\infty \frac{1}{\sqrt{n}}$ $\quad\left[p = \frac{1}{2} \leq 1\right]$.

11. $\displaystyle 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^\infty \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ are convergent p-series with $p = 4 > 1$ and $p = \frac{3}{2} > 1$, respectively. Thus,

$\displaystyle\sum_{n=1}^\infty\left(\frac{5}{n^4} + \frac{4}{n\sqrt{n}}\right) = 5\sum_{n=1}^\infty \frac{1}{n^4} + 4\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ is convergent by Theorems 8.2.8(i) and 8.2.8(ii).

13. $f(x) = xe^{-x}$ is continuous and positive on $[1,\infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1,\infty)$. Thus, the Integral Test applies.

$$\begin{aligned}\int_1^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \quad \text{[by parts]}\\ &= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e\end{aligned}$$

since $\lim\limits_{b\to\infty} be^{-b} = \lim\limits_{b\to\infty}(b/e^b) \overset{\text{H}}{=} \lim\limits_{b\to\infty}(1/e^b) = 0$ and $\lim\limits_{b\to\infty} e^{-b} = 0$. Thus, $\displaystyle\sum_{n=1}^\infty ne^{-n}$ converges.

14. $\displaystyle f(x) = \frac{x^2}{x^3+1}$ is continuous and positive on $[2,\infty)$, and also decreasing since $\displaystyle f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \geq 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1,\infty)$].

$\displaystyle\int_2^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{3}\ln(x^3+1)\right]_2^t = \frac{1}{3}\lim_{t\to\infty}\left[\ln(t^3+1) - \ln 9\right] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^3+1}$.

Another solution: Use the Limit Comparison Test with $\displaystyle a_n = \frac{n^2}{n^3+1}$ and $\displaystyle b_n = \frac{1}{n}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2\cdot n}{n^3+1} = \lim_{n\to\infty}\frac{1}{1+1/n^3} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^3+1}$.

5. $f(x) = \dfrac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1 + \ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ diverges.

6. $\dfrac{n^2 - 1}{3n^4 + 1} < \dfrac{n^2}{3n^4 + 1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\dfrac{1}{n^2}$. $\displaystyle\sum_{n=1}^\infty \frac{n^2 - 1}{3n^4 + 1}$ converges by comparison with $\displaystyle\sum_{n=1}^\infty \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series $[p = 2 > 1]$. The terms of the given series are positive for $n > 1$, which is good enough.

. $\dfrac{\cos^2 n}{n^2 + 1} \le \dfrac{1}{n^2 + 1} < \dfrac{1}{n^2}$, so the series $\displaystyle\sum_{n=1}^\infty \frac{\cos^2 n}{n^2 + 1}$ converges by comparison with the p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ $[p = 2 > 1]$.

. $\dfrac{4 + 3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^\infty \frac{4 + 3^n}{2^n}$ diverges by comparison with the divergent geometric series $\displaystyle\sum_{n=1}^\infty \left(\tfrac{3}{2}\right)^n$.

. $\dfrac{n - 1}{n\,4^n}$ is positive for $n > 1$ and $\dfrac{n - 1}{n\,4^n} < \dfrac{n}{n\,4^n} = \dfrac{1}{4^n} = \left(\dfrac{1}{4}\right)^n$, so $\displaystyle\sum_{n=1}^\infty \frac{n - 1}{n\,4^n}$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^\infty \left(\frac{1}{4}\right)^n$.

$\dfrac{1}{\sqrt{n^3 + 1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n^3 + 1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ $\left[p = \tfrac{3}{2} > 1\right]$.

Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2 + 1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + (1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n^2 + 1}}$.

Use the Limit Comparison Test with $a_n = \dfrac{1}{2n + 3}$ and $b_n = \dfrac{1}{n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n + 3} = \lim_{n\to\infty} \frac{1}{2 + (3/n)} = \frac{1}{2} > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^\infty \frac{1}{2n + 3}$.

$\dfrac{2 + (-1)^n}{n\sqrt{n}} \le \dfrac{3}{n\sqrt{n}}$, and $\displaystyle\sum_{n=1}^\infty \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}}$ $\left[p = \tfrac{3}{2} > 1\right]$, so the given series converges by the Comparison Test.

$\dfrac{1 + \sin n}{10^n} \le \dfrac{2}{10^n}$ and $\displaystyle\sum_{n=0}^\infty \frac{2}{10^n} = 2\sum_{n=0}^\infty \left(\frac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a onvergent geometric series.

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln n = \lim_{x\to\infty} \ln x = \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by part (a).

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \le a_n < 1$ for all $n > N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

$b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}$. This is a p-series, which converges for all b such that $-\ln b > 1 \Leftrightarrow$ $\ln b < -1 \Leftrightarrow b < e^{-1} \Leftrightarrow b < 1/e$ [with $b > 0$].

Other Convergence Tests

(a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

(a) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

$\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{n+6}$. Now $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

$-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$. Here $a_n = (-1)^n \frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

$b_n = \frac{1}{\sqrt{n}} > 0$. $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

$\sum a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1 + 2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2 + 1/\sqrt{n}} = \frac{1}{2} \ne 0$. Since $\lim_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

$\sum a_n = \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3 - 1/n}{2 + 1/n} = \frac{3}{2} \ne 0$. Since $\lim_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

9. The series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!} = \sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test because

$b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2\cdot 2^n}{(n+1)n!} = \frac{2}{n+1}\cdot\frac{2^n}{n!} = \frac{2}{n+1}\cdot b_n \leq b_n$ and (ii) $\lim_{n\to\infty}\frac{2^n}{n!} = \frac{2}{n}\cdot\frac{2}{n-1}\cdot\cdots\cdot\frac{2}{2}\cdot\frac{2}{1} = 0$, series is convergent. Now $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms tc sum to the desired accuracy.)

10. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,5^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)5^{n+1}} < \frac{1}{n\,5^n}$ and (ii) $\lim_{n\to\infty}\frac{1}{n\,5^n} =$ the series is convergent. Now $b_4 = \frac{1}{4\cdot 5^4} = 0.0004 > 0.0001$ and $b_5 = \frac{1}{5\cdot 5^5} = 0.000064 < 0.0001$, so by the Alter Series Estimation Theorem, $n = 4$. (That is, since the 5th term is less than the desired error, we need to add the first 4 te get the sum to the desired accuracy.)

11. The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^6} < \frac{1}{n^6}$ and (ii) $\lim_{n\to\infty}\frac{1}{n^6} = 0$, so th series is convergent. Now $b_5 = \frac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \frac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to g sum to the desired accuracy.)

12. Using the Ratio Test with the series $\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{e^n}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)}{e^{n+1}}\cdot\frac{e^n}{(-1)^{n-1}n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^1(n+1)}{en}\right| = \frac{1}{e}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{e}(1) = \frac{1}{e} < 1,$$

so the series is absolutely convergent (and therefore convergent). Now $b_6 = 6/e^6 \approx 0.015 > 0.01$ and $b_7 = 7/e^7 \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 6$. (That is, since the 7th term is less tl the desired error, we need to add the first 6 terms to get the sum to the desired accuracy.)

13. $b_7 = \frac{1}{7^5} = \frac{1}{16{,}807} \approx 0.000\,059\,5$, so

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5} \approx s_6 = \sum_{n=1}^{6}\frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.972\,080$. Adding b_7 to s_6 does not cha the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.9721.

14. $b_6 = \frac{6}{8^6} = \frac{6}{262{,}144} \approx 0.000\,023$, so $\sum_{n=1}^{\infty}\frac{(-1)^n n}{8^n} \approx s_5 = \sum_{n=1}^{5}\frac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32{,}768} \approx -0.098\,7$ Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.0988.

$b_7 = \dfrac{7^2}{10^7} = 0.000\,004\,9$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n-1} n^2}{10^n} = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10{,}000} + \frac{25}{100{,}000} - \frac{36}{1{,}000{,}000} = 0.067\,614$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.0676.

$b_6 = \dfrac{1}{3^6 \cdot 6!} = \dfrac{1}{524{,}880} \approx 0.000\,001\,9$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n}{3^n n!} = -\frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29{,}160} \approx -0.283\,471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.2835.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

If $p > 0$, $\dfrac{1}{(n+1)^p} \le \dfrac{1}{n^p}$ $[\{1/n^p\}$ is decreasing$]$ and $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

Using the Ratio Test, $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}/(n+1)^3}{(-3)^n/n^3}\right| = \lim_{n\to\infty} \left|\frac{(-3)n^3}{(n+1)^3}\right| = 3 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 3 > 1$, so the series diverges.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has positive terms and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right] = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

$\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Using the Ratio Test, $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| = \lim_{n\to\infty} \left|\frac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series: $\displaystyle\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. But $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges by the Alternating Series Test: $\left\{\dfrac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\dfrac{x}{x^2+1}\right)' = \dfrac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$, and $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Thus, $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is conditionally convergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a divergent p-series $\left[p = \frac{1}{4} \le 1\right]$, so the given series is conditionally convergent.

24. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4}=\infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

25. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+2)4^{2(n+1)+1}}\cdot\frac{(n+1)4^{2n+1}}{10^n}\right]=\lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+2)4^{2n+3}}\cdot\frac{(n+1)4^{2n+1}}{10^n}\right]$

$$=\lim_{n\to\infty}\left(\frac{10}{4^2}\cdot\frac{n+1}{n+2}\right)=\frac{5}{8}<1,$$

so the series is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergen same as convergence.

26. $\left|\frac{\sin 4n}{4^n}\right|\le\frac{1}{4^n}$, so $\sum_{n=1}^{\infty}\left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{4^n}$ $\left[|r|=\frac{1}{4}\right.$.
Thus, $\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$ is absolutely convergent.

27. $\frac{|\cos(n\pi/3)|}{n!}\le\frac{1}{n!}$ and $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (use the Ratio Test), so the series $\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$ converges absolutely by Comparison Test.

28. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{5^n/[(n+2)^2\,4^{n+3}]}{5^{n-1}/[(n+1)^2\,4^{n+2}]}=\frac{5}{4}\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2=\frac{5}{4}>1$, so the series diverges by the Ratio Te

29. $\left|\frac{(-1)^n\arctan n}{n^2}\right|<\frac{\pi/2}{n^2}$, so since $\sum_{n=1}^{\infty}\frac{\pi/2}{n^2}=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges $[p=2>1]$, the given series $\sum_{n=1}^{\infty}\frac{(-1)^n\,\text{arct}}{n^2}$ converges absolutely by the Comparison Test.

30. $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{1}{\ln n}=0<1$, so the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

31. $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left(\frac{n^n}{3^{1+3n}}\right)^{1/n}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{3}\cdot 3^3}=\infty$, so the series $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{3^{4+3n}}\cdot\frac{3^{1+3n}}{n^n}\right]=\lim_{n\to\infty}\left[\frac{1}{3^3}\cdot\left(\frac{n+1}{n}\right)^n(n+1)\right]$

$$=\frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1)=\tfrac{1}{27}e\lim_{n\to\infty}(n+1)=\infty,$$

so the series is divergent by the Ratio Test.

32. Since $\left\{\frac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n\ln n}=0$, the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Sinc $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test (Exercise 8.3.15), the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is conditionally convergent.

33. $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}=\lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2}=\frac{1}{2}<1$, so the series $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by tl Root Test.

34. $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{1}{\arctan n}=\frac{1}{\pi/2}=\frac{2}{\pi}<1$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Tes