the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \dfrac{x^n}{\sqrt{n}}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim_{n\to\infty}\frac{|x|}{\sqrt{1+1/n}} = |x|$.

By the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} \le 1$. When $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

5. If $a_n = \dfrac{(-1)^{n-1}x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)xn^3}{(n+1)^3}\right| = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^3|x|\right] = 1^3\cdot|x| = |x|$$

converges by the Alternating Series Test. When $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^3} = -\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series $[p = 3 > 1]$. Thus, the interval of convergence is $I = [-1, 1]$.

6. $a_n = \sqrt{n}\,x^n$, so we need $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\,|x| = |x| < 1$ for convergence (by the Ratio Test), so $R = 1$. When $x = \pm 1$, $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\sqrt{n} = \infty$, so the series diverges by the Test for Divergence. Thus, $I = (-1, 1)$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$, and $I = (-\infty, \infty)$.

8. If $a_n = \dfrac{x^n}{n3^n}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{xn}{(n+1)3}\right| = \frac{|x|}{3}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x|}{3}$. By the Ratio Test, the series converges when $\dfrac{|x|}{3} < 1 \;\Leftrightarrow\; |x| < 3$, so $R = 3$. When $x = -3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, $I = [-3, 3)$.

$\left[\text{so } R = \frac{1}{2}\right] \Leftrightarrow -\frac{7}{2} < x < -\frac{5}{2}$. When $x = -\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$. When $x = -\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{7}{2}, -\frac{5}{2}\right]$.

15. $a_n = \frac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\frac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b$ $[\text{so } R = b] \Leftrightarrow -b < x-a < b \Leftrightarrow a-b < x < a+b$. When $|x-a| = b$, $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

16. $a_n = \frac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n\,|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}\,|x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x-4| < 1$ $[\text{so } R = 1] \Leftrightarrow -1 < x-4 < 1 \Leftrightarrow 3 < x < 5$. When $|x-4| = 1$, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$, which converges by comparison with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $[p = 2 > 1]$. Thus, $I = [3, 5]$.

17. If $a_n = n!(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n \to \infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

18. $a_n = \frac{n^2x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \frac{n^2x^n}{2^n n!} = \frac{nx^n}{2^n(n-1)!}$, so

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\frac{2^n(n-1)!}{n\,|x|^n} = \lim_{n\to\infty}\frac{n+1}{n^2}\frac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

19. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \leq 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

20. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

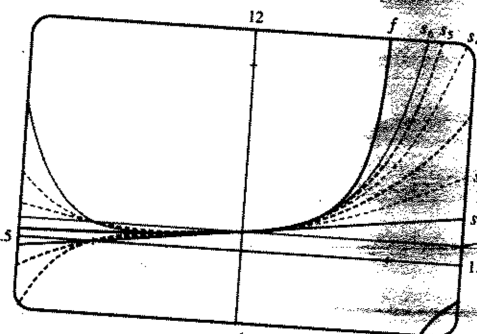
21. If $a_n = \dfrac{(n!)^k}{(kn)!}x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^k\,(kn)!}{(n!)^k\,[k(n+1)]!}|x| = \lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}|x|$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x| = \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x|$$

$$= \left(\frac{1}{k}\right)^k |x| < 1 \quad\Leftrightarrow\quad |x| < k^k \text{ for convergence, and the radius of convergence is } R = k^k.$$

22. The partial sums of the series $\sum_{n=0}^{\infty} x^n$ definitely do not converge to $f(x) = 1/(1-x)$ for $x \geq 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \leq -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \geq 1$ (see Example 8.2.5).

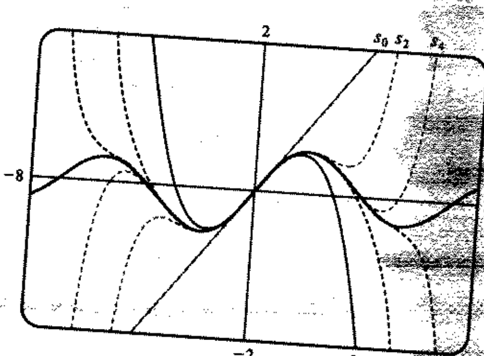


23. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}}\cdot\frac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0 \text{ for all } x.$$

So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

(b), (c) The initial terms of $J_1(x)$ up to $n = 5$ are $a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$, $a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



24. (a) $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)(3n)}$, so $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|^3\lim_{n\to\infty}\dfrac{1}{(3n+2)(3n+3)}$ for all x, so the domain is $\mathbb{R}$.

27. We use the Root Test on the series $\sum c_n x^n$. We need $\lim_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \lim_{n\to\infty} \sqrt[n]{|c_n|} = c|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

28. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n (x-a)^n$, we find that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty} \frac{|x-a|}{|c_n/c_{n+1}|} \ (*) \ = \frac{|x-a|}{\lim_{n\to\infty} |c_n/c_{n+1}|} \quad \left[\text{if } \lim_{n\to\infty} |c_n/c_{n+1}| \neq 0\right],$$

so the series converges when $\dfrac{|x-a|}{\lim_{n\to\infty} |c_n/c_{n+1}|} < 1 \iff |x-a| < \lim_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$. Thus, $R = \lim_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$. If $\lim_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right| = 0$ and $|x-a| \neq 0$, then $(*)$ shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$.

29. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 8.2.43, $\sum (c_n + d_n)\, x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)\, x^n$ is 2.

30. Since $\sum c_n x^n$ converges whenever $|x| < R$, $\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges whenever $|x^2| < R \iff |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

8.6 Representing Functions as Power Series

1. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum_{n=0}^{\infty} \dfrac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \iff |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{3}{1-x^4} = 3\left(\dfrac{1}{1-x^4}\right) = 3(1 + x^4 + x^8 + x^{12} + \cdots) = 3\sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$ with $|x^4| < 1 \iff |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

[Note that $3\sum_{n=0}^{\infty} (x^4)^n$ converges $\iff$ $\sum_{n=0}^{\infty} (x^4)^n$ converges, so the appropriate condition [from Equation (1)] is $|x^4| < 1$.]

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$. The series converges when $|x^3| < 1 \iff |x|^3 < 1 \iff |x| < \sqrt[3]{1} \iff |x| < 1$. Thus, $R = 1$ and $I = (-1, 1)$.

6. $f(x) = \dfrac{1}{1+9x^2} = \dfrac{1}{1-(-9x^2)} = \sum\limits_{n=0}^{\infty}(-9x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n\, 3^{2n}\, x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

7. $f(x) = \dfrac{1}{x-5} = -\dfrac{1}{5}\left(\dfrac{1}{1-x/5}\right) = -\dfrac{1}{5}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n$ or equivalently, $-\sum\limits_{n=0}^{\infty}\dfrac{1}{5^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

8. $f(x) = \dfrac{x}{4x+1} = x \cdot \dfrac{1}{1-(-4x)} = x\sum\limits_{n=0}^{\infty}(-4x)^n = \sum\limits_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

9. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n}$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}$$

The geometric series $\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \;\Leftrightarrow\; \dfrac{|x^2|}{9} < 1 \;\Leftrightarrow\; |x|^2 < 9 \;\Leftrightarrow\; |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

10. $f(x) = \dfrac{x^2}{a^3 - x^3} = \dfrac{x^2}{a^3} \cdot \dfrac{1}{1 - x^3/a^3} = \dfrac{x^2}{a^3}\sum\limits_{n=0}^{\infty}\left(\dfrac{x^3}{a^3}\right)^n = \sum\limits_{n=0}^{\infty}\dfrac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \;\Leftrightarrow\; |x^3| < |a^3| \;\Leftrightarrow\; |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

11. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1} \;\Rightarrow\; 3 = A(x-1) + B(x+2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\frac{3}{x^2+x-2} = \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[-1 - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\tfrac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

12. $f(x) = \dfrac{7x-1}{3x^2+2x-1} = \dfrac{7x-1}{(3x-1)(x+1)} = \dfrac{A}{3x-1} + \dfrac{B}{x+1} = \dfrac{1}{3x-1} + \dfrac{2}{x+1} = 2 \cdot \dfrac{1}{1-(-x)} - \dfrac{1}{1-3x}$

$$= 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left[2(-1)^n - 3^n\right]x^n$$

The series $\sum(-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum(3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum\limits_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$f(x) = \ln(1+x) = \int \dfrac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}$

[$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

15. $f(x) = \ln(5-x) = -\displaystyle\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,5^n}$.

Putting $x = 0$, we get $C = \ln 5$ The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

16. We know that $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum_{n=1}^{\infty}2^n nx^{n-1} = \sum_{n=0}^{\infty}2^{n+1}(n+1)x^n$, so

$f(x) = \dfrac{x^2}{(1-2x)^2} = \dfrac{x^2}{2}\cdot\dfrac{2}{(1-2x)^2} = \dfrac{x^2}{2}\sum_{n=0}^{\infty}2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty}2^n(n+1)x^{n+2}$ or $\sum_{n=2}^{\infty}2^{n-2}(n-1)x^n$,

with $R = \frac{1}{2}$.

17. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n$ for $\left|\dfrac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. Now

$\dfrac{1}{(x-2)^2} = \dfrac{d}{dx}\left(\dfrac{1}{2-x}\right) = \dfrac{d}{dx}\left(\sum_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n\right) = \sum_{n=1}^{\infty}\dfrac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n$. So

$f(x) = \dfrac{x^3}{(x-2)^2} = x^3\sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^{n+3}$ or $\sum_{n=3}^{\infty}\dfrac{n-2}{2^{n-1}}x^n$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$. Thus,

$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty}(-1)^n\dfrac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\dfrac{1}{3^{2n+1}(2n+1)}x^{2n+1}$ for $\left|\dfrac{x}{3}\right| < 1 \Leftrightarrow |x| < 3$, so $R = 3$.