Homework 10 Solutions

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(18) This is an upper triangular matrix, and so is characteristic polynomial $f_A(\lambda)$ is equal to $\prod_{i=1}^{4}(\lambda - a_{ii})$. In this case, the eigenvalues are 0 and 1, both with algebraic multiplicity two. The space E_0 is equal to the kernel of A, which is easily seen to be $\text{span}\{\hat{e}_1, \hat{e}_3\}$. The space E_1 corresponds to the kernel of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is given by span $\hat{e}_2$. There is clearly no eigenbasis.

(20) The eigenspace E_1 corresponds to the kernel of the matrix

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

If $a = 0$, the geometric multiplicity of E_1 is two, because you have two columns that are both zero. Otherwise, the second and third column vectors are linearly independent, and so in this case the geometric multiplicity is one. The eigenvalue 2 has algebraic multiplicity one, so the dimension of E_2 is one no matter what a, b, c are.

(24) This problem is difficult. Here is one solution, though I suspect a better one exists. Let A be a matrix that has $E_1 = \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as its only eigenspace. Then $f_A(\lambda)$ has two real roots (with multiplicities) since it is quadratic and has at least one real root. Both of these roots must be equal to 1, otherwise A would have two linearly independent eigenvectors, which we are told it does not. Therefore $f_A(\lambda) = (\lambda - 1)^2$. Consider the basis

$$\mathcal{B} = \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 \},$$

where $\vec{v}_2$ is arbitrary except for the fact that it is not a scalar multiple of [2 1]. By the properties of A, in this basis A has the representation

$$A' = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}.$$

Since $f_A(\lambda)$ does not depend on the basis, we see immediately that $b = 1$. A quick calculation shows that A' has only one eigenspace, namely span [1 0], iff $a \neq 0$.

The punchline - the way that we have chosen $\mathcal{B}$ and A' guarantees that A will have the properties we want. Why is this? Suppose that two $n \times n$ matrices T and L are similar, and let $\vec{v}$ be an eigenvector of L. Then if $T = S^{-1}LS$, $S^{-1}\vec{v}$ is an eigenvector of T with the same eigenvalue, i.e. the eigenspaces of T are in one-to-one correspondence with the eigenspaces of L via the transformation $\vec{v} \to S^{-1}\vec{v}$. Applying this to our situation, A' has as its unique eigenspace [1 0] (represented in the basis $\mathcal{B}$), which corresponds to [2 1] via the transformation $\mathcal{B} \mapsto B$, B=standard basis, and by what we've just said this guarantees that A will have [2 1] as its unique eigenspace with eigenvalue 1 as long as $a \neq 0$. Choosing $\vec{v}_2 = [0\ 1]$, $a = 1$, and performing the basis change $\mathcal{B} \to B$, we find such an A to be

$$\begin{bmatrix} 0 & 2 \\ -\frac{1}{2} & 2 \end{bmatrix}.$$

(26) $f_A(\lambda) = \lambda^6 + \cdots + \det A$, so that $f_A \to \infty$ as $\lambda \to \infty$. Clearly $f_A(0) = \det A$. So $\det A < 0$ implies that there must be at least one positive real root by the intermediate value theorem.