110.201 HW 12 Solutions

May 9, 2005

pp. 370-372: 6, 12, 16.

(6) The characteristic polynomial of A is given by $f_A(\lambda) = \lambda(\lambda^2 - 9)$, so that the eigenvalues of A are 0, ± 3. The geometric multiplicity of each eigenvalue is one, and so by the spectral theorem, $E_0, E_3 and E_{-3}$ are all orthogonal to one another. Thus when we find the eigenvectors, all we have to do is normalize them. DO NOT USE G-S! It is unnecessary. A quick calculation shows that

$$E_0 = \text{span}\{\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}\},\ E_3 = \text{span}\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\},\ E_{-3} = \text{span}\{\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}\}$$

.

Normalize the vectors above to get an orthonormal eigenbasis.

(12) We recall that $\text{ref}_L\vec{w} = \vec{w} - 2(\vec{w})^\perp$, where $\vec{w}^\perp$ is the part of $\vec{w}$ orthogonal to L. Thus $\text{ref}_L(\vec{w}) = -\vec{w}$ for any $\vec{w}$ orthogonal to L. Therefore to find an orthonormal eigenbasis for this transformation, it suffices to find two orthogonal unit vectors perpendicular to L, and a unit vector that spans L. These vectors, by trial and error (or Graham - Schmidt if you like) are given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix},\ \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

The first two are orthogonal to L, and the third spans L. The matrix for the transformation with respect to this basis is $\text{diag}(-1, -1, 1)$. In the standard basis, it is, by quick computation,

$$\begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & 3/5 \end{pmatrix}$$

(16) OK, you can do this the hard way or the easy way. Naturally, I choose to do it the easy way. First, it is clear that the kernel of A is four-dimensional, thus $f_A(\lambda) = \lambda^4(\lambda - \alpha)$ for some α. Now notice that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

so that $\alpha = 5$. Why do the algebraic multiplicities agree with the geometric multiplicities? Because the kernel of A is four-dimensional. If you want a deeper reason, here's one: The image of A is one-dimensional, because all the columns of A are the same. Thus the kernel of A^t is four-dimensional. But $A^t = A$.