HW 13 Solutions: pp 389-391 4, 6, 8.

May 11, 2005

(4) We compute

\[ B \equiv A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \]

From here \( f_B(\lambda) = \lambda^2 - 3\lambda + 1 \), so the eigenvalues of \( B \) are

\[ \lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}. \]

Thus the singular values of \( A \) are the positive square roots of the above eigenvalues:

\[ \sigma_1 = \sqrt{\lambda_+}, \quad \sigma_2 = \sqrt{\lambda_-}. \]

(6) As in (4), we compute

\[ B \equiv A^T A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \]

The eigenvalues of \( B \) are then (skipping the calculation)

\[ \lambda = 0 \quad \lambda = 25. \]

Therefore, with the convention of listing singular values in decreasing order, \( \sigma_1 = \sqrt{25} = 5 \) and \( \sigma_2 = 0 \).

Using the ideas of section 8.3, we first find an orthonormal eigenbasis of \( B = A^T A \):

\[ E_0 = \ker(B) = \text{span} \ \hat{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \]

\[ E_{25} = \ker(B - 25I) = \text{span} \ \hat{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]
Therefore $\|A\hat{v}_1\| = 5 = \sigma_1$. The unit circle $T$ consists of all points of the form $(t \in [0, 2\pi])$
\[
\cos(t)\hat{v}_1 + \sin(t)\hat{v}_2.
\]
Under $A$, $\hat{v}_2 \to 0$, and $\hat{v}_1 \to \sqrt{5}\hat{v}_1$. Thus the image of the unit circle under $A$ is the line segment joining the points

$$p_{\pm} = \pm\sqrt{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

(8) In this case we have a $2 \times 2$ matrix, so the matrices $U, \Sigma$ and $V$ appearing in the SVD are all $2 \times 2$. The algorithm for singular value decomposition tells us that

$$A = U\Sigma V^T,$$

where $V = $ matrix with columns the orthonormal eigenvectors of $A^TA$, $\Sigma = $ diagonal matrix with entries the singular values of $A$, and $U = $ orthonormal basis of $\mathbb{R}^2$ with columns the vectors $\sigma_i^{-1}A\hat{v}_i$. We compute

$$A^TA = \begin{bmatrix} p^2 + q^2 & 0 \\ 0 & p^2 + q^2 \end{bmatrix}.$$ 

$A^TA$ thus has $\hat{e}_1$ and $\hat{e}_2$ as orthonormal eigenvectors, and $A$ has singular values

$$\sigma_{1,2} = \sqrt{p^2 + q^2}.$$ 

We can now read off

$$\Sigma = \begin{bmatrix} \sqrt{p^2 + q^2} & 0 \\ 0 & \sqrt{p^2 + q^2} \end{bmatrix},$$ 

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$ 

$$U = \frac{1}{\sqrt{p^2 + q^2}} \begin{bmatrix} p & -q \\ q & p \end{bmatrix}.$$ 

Naturally, most of this was unnecessary, since $A$ is itself a scalar multiple of an orthogonal matrix. But this is how the process goes in general.