HW 13 Solutions: pp 389-391 4, 6, 8.

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(4) We compute

$$B \equiv A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

¿From here $f_B(\lambda) = \lambda^2 - 3\lambda + 1$, so the eigenvalues of B are

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}.$$

Thus the singular values of A are the positive square roots of the above eigenvalues:

$$\sigma_1 = \sqrt{\lambda_+}, \quad \sigma_2 = \sqrt{\lambda_-}.$$

(6) As in (4), we compute

$$B \equiv A^T A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

The eigenvalues of B are then (skipping the calculation)

$$\lambda = 0 \quad \lambda = 25.$$

Therefore, with the convention of listing singular values in decreasing order, $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = 0$.
Using the ideas of section 8.3, we first find an orthonormal eigenbasis of $B = A^T A$:

$$E_0 = \ker(B) = \text{span } \hat{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$E_{25} = \ker(B - 25I) = \text{span } \hat{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore $\|A\hat{v}_1\| = 5 = \sigma_1$. The unit circle T consists of all points of the form $(t \in [0, 2\pi))$

$$\cos(t)\hat{v}_1 + \sin(t)\hat{v}_2.$$

Under A, $\hat{v}_2 \to 0$, and $\hat{v}_1 \to \sqrt{5}\hat{v}_1$. Thus the image of the unit circle under A is the line segment joining the points

$$p_\pm = \pm\sqrt{5}\begin{bmatrix}1\\2\end{bmatrix}.$$

(8) In this case we have a 2×2 matrix, so the matrices U, Σ and V appearing in the SVD are all 2×2. The algorithm for singular value decomposition tells us that

$$A = U\Sigma V^T,$$

where $V =$ matrix with columns the orthonormal eigenvectors of A^TA, $\Sigma =$ diagonal matrix with entries the singular values of A, and $U =$ orthonormal basis of $\mathbb{R}^2$ with columns the vectors $\sigma_i^{-1}A\vec{v}_i$. We compute

$$A^TA = \begin{bmatrix}p^2+q^2 & 0\\ 0 & p^2+q^2\end{bmatrix}.$$

A^TA thus has $\hat{e}_1$ and $\hat{e}_2$ as orthonormal eigenvectors, and A has singular values

$$\sigma_{1,2} = \sqrt{p^2+q^2}.$$

We can now read off

$$\Sigma = \begin{bmatrix}\sqrt{p^2+q^2} & 0\\ 0 & \sqrt{p^2+q^2}\end{bmatrix},$$

$$V = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$U = \frac{1}{\sqrt{p^2+q^2}}\begin{bmatrix}p & -q\\ q & p\end{bmatrix}.$$

Naturally, most of this was unnecessary, since A is itself a scalar multiple of an orthogonal matrix. But this is how the process goes in general.