110.201 Homework 4 Solutions

March 7, 2005

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(22) Recall the fundamental relation dim image(A) = rk(A). Therefore the image will be a point, line, plane or all of $\mathbb{R}^3$ depending on whether the matrix A has rank 0, 1, 2, or 3, respectively. To compute the rank, find the reduced row echelon form of A, which is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus A has rank 2, and the image is a plane.
There are other approaches to this problem. A popular (and quick, if done right) one was to determine the number of linearly independent columns of A. From class, it is true that the dimension of the image of A is the dimension of the span of the columns of A.

(32) So many choices! The simplest, of course, is to just let $T : \mathbb{R} \to \mathbb{R}^3$ be given by

$$T(x) = x \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

Strangely, this was not the most popular solution; that prize went to projecting all of $\mathbb{R}^3$ onto the given line. Either is acceptable.

(42) A common misconception encountered while grading: the statement $\ker(A) = V$, where A is some linear transformation and V a vector space, means that V is *the whole* kernel, not just contained in it. For example, in

this problem it is not acceptable to just let B be the zero matrix so that Image(A) is trivially contained in the kernel of B.
Following the book, set up the problem like so. The image of A is the set of all vectors

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

such that the system of equations $A\vec{x} = \vec{y}$ has a solution. Following the usual procedures for finding solutions to such equations, we form an augmented matrix for the system and get it into reduced row echelon form. What we find is

$$x_1 - x_3 + 8x_4 = 4y_3 - 3y_4$$

$$x_2 + 2x_3 - 2x_4 = -y_3 + y_4$$

$$0 = y_1 - 3y_3 + 2y_4$$

$$0 = y_2 - 2y_3 + y_4$$

The top two equations form an over-determined system, and so the whole system will have a solution iff the bottom two equations are satisfied. It follows that the image of A is the set of all $\vec{y} \in \mathbb{R}^4$ such that

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \vec{y} = 0.$$

All that was done was to write the last two equations in matrix form. This gives us the desired matrix B.

(44) (a) This is true. To prove it, recall from the first weeks of class that solutions to the homogeneous equation $A\vec{x} = 0$ were not changed by elementary row operations. More explicitly, if we have a solution to the homogeneous system of linear equations

$$\begin{array}{ccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0 \end{array}$$

then it is also a solution of the reduced system.

(b) This is *not* true! Counterexamples abound; a simple one is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The image of A is the line spanned by $(1, 2)$ and that of rref(A) is the line spanned by $(1, 0)$.
This one confused a great many students. What is going on is this: Consider the equation

$$A\vec{x} = \vec{y}$$

Finding the kernel of A is about solving for $\vec{x}$ when $\vec{y} = 0$. As we've discussed, these solutions don't change when we do row operations. However, finding the image of A is about finding the span of the columns of A. Since the columns of A can change drastically under row operations, so can this span. For example, in the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

switching the rows amounts to changing the span of the columns from the line $t(1, 0)$ to the line $t(0, 1)$, t a real number.