110.201 Linear Algebra
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March 7, 2005

2. If W is a subspace, we know that it must be closed under scalar multiplication. However, for any vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we have $x \leq y \leq z$. However, multiplying each quantity by -1 we obtain the relation $-x \geq -y \geq -z$, which is definitely not in W. So W fails to be closed under scalar multiplication. Alternatively, note that W is neither $\vec{0}$, a line through $\vec{0}$, a plane through $\vec{0}$, or $\mathbb{R}^3$ itself.

6a. Let V and W be subspaces of $\mathbb{R}^n$. We use properties of V and W separately to show that $V \cap W$ is a subspace. So, if $V \cap W$ is a subspace of $\mathbb{R}^n$, then

1. It must contain $\vec{0}$. But $\vec{0} \in V \cap W$ since V and W are each subspaces, so $\vec{0} \in V$ and $\vec{0} \in W$.
2. It must be closed under addition. Let $\vec{x}, \vec{y} \in V \cap W$. Then, in particular, $\vec{x}, \vec{y} \in V$, so that $\vec{x}+\vec{y} \in V$. Similarly, $\vec{x}+\vec{y} \in W$. Therevore, $\vec{x}+\vec{y} \in V \cap W$.
3. It must be closed under scalar multiplication. Let k be a scalar and let $\vec{x} \in V \cap W$. Then $k\vec{x} \in V$ and $k\vec{x} \in W$. Hence $k\vec{x} \in V \cap W$.

6b. To show that this is false, we only need to find a counterexample. So, for instance, let $n = 2$, and let V be the x-axis and W be the y-axis. Then, $\vec{e_1} \in V$ and $\vec{e_2} \in W$, but $\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V \cup W$, so $V \cup W$ fails to be closed under addition. Of course, there are many other counterexamples to be found.

36. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m} \in \mathbb{R}^n$, are linearly dependent, then there exists some nontrivial relation

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots c_m\vec{v_m} = \vec{0}$$

with $c_1, c_2, \ldots, c_m$ not all zero. Now apply the linear transformation T:

$$T(c_1\vec{v_1} + c_2\vec{v_2} + \ldots c_m\vec{v_m}) = T(\vec{0}) = \vec{0},$$

so that

$$c_1T(\vec{v_1}) + c_2T(\vec{v_2}) + \dots c_mT(\vec{v_m}) = \vec{0}.$$

But this is a nontrivial relation between $T(\vec{v_1}), T(\vec{v_2}), \dots, T(\vec{v_m})$, so they must be linearly dependent.

48. Note that there are no restrictions on the sizes of the matrices A and B, so that there are many possible solutions to this problem. For instance,

$$3x_1 + 4x_2 + 5x_3 = [3\ 4\ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

so that $V = \ker[3\ 4\ 5]$. Next, to express V as the image of some matrix, choose a basis of V, such as $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix} \right\}$. Then $V = \operatorname{im} \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$.